On the General Bahadur-Kiefer, Quantile, and Vervaat Processes: Old and New

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Abstract

Adopting E. Parzen's ideas of 1979, we assume that the density-quantile function $f \circ F^{-1}$ is continuous on $(0,1)$ and regularly varying in neighbourhoods of 0 and 1. Under this assumption we then prove strong and weak limit theorems for the general Bahadur-Kiefer and quantile processes. In particular, our investigations provide a new look at results obtained by G.R. Shorack in 1972, as well as throw a new light on those previously proved under a condition introduced by M. Csörgő and P. Révész in 1978. The problem of constructing asymptotic confidence bands for the general quantile function F^{-1} is also discussed in detail, and in a historical context as well. Furthermore, the above mentioned results concerning the general Bahadur-Kiefer and quantile processes play a decisive role when investigating the asymptotic behaviour of the general Vervaat process V_n. The herein obtained strong and weak convergence results for the process V_n supplement and generalize the only known results so far in the area that were obtained by W. Vervaat in 1972.

1 Introduction and some results

Let X be a random variable with distribution function F. The quantile function F^{-1} of F is defined as follows

$$F^{-1}(t) := \inf\{x : \ F(x) \geq t\}, \quad 0 < t < 1.$$

Let $X_1, ..., X_n$ be independent copies of X, and let F_n denote the empirical distribution function

$$F_n(x) := \frac{1}{n} \sum_{k=1}^{n} \mathbf{I}\{X_k \leq x\}, \quad -\infty < x < \infty,$$

where $\mathbf{I}$ denotes the indicator function. The corresponding (general) empirical and quantile processes are defined, respectively, as follows

$$\beta_n(x) := F_n(x) - F(x), \quad -\infty < x < \infty, \tag{1.1}$$

$$\gamma_n(t) := F_n^{-1}(t) - F^{-1}(t), \quad 0 < t < 1, \tag{1.2}$$

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where F_n^{-1} is the left-continuous inverse $F_n^{-1}(x) := \inf\{x : F_n(x) \geq t\}$ of F_n.

Remark 1.1 Traditionally, it is the normalized versions $\sqrt{n}\beta_n$ and $\sqrt{n}\gamma_n$ that are called, respectively, the empirical and quantile processes. We do not follow this tradition in the current paper for the sake of avoiding re-normalizations of already normalized processes. Thus, in the current paper, we will normalize all the processes only once, when needed. □

Throughout the paper we assume that the distribution function F is continuous. Therefore, the random variable $U := F(X)$ is $(0,1)$-uniform, and so are also the following independent random variables $U_1 := F(X_1)$, ..., $U_n := F(X_n)$. The corresponding $(0,1)$-uniform empirical and quantile processes are defined, respectively, as follows

$$\beta_n^U(t) := E_n(t) - t, \quad 0 \leq t \leq 1, \tag{1.3}$$

$$\gamma_n^U(t) := E_n^{-1}(t) - t, \quad 0 \leq t \leq 1, \tag{1.4}$$

where E_n is the empirical distribution function corresponding to $U_1, \ldots, U_n$, and E_n^{-1} is its left-continuous inverse.

An application of the Taylor formula suggests that the balanced quantile process $f \circ F^{-1}\gamma_n$ should asymptotically behave like the $(0,1)$-uniform quantile process γ_n^U. M. Csörgő and Révész (1978), in combination with M. Csörgő, S. Csörgő, Horváth and Révész [CsCsHR] (1984), established the following theorem.

Theorem 1.1 [M. Csörgő and Révész, 1978, CsCsHR, 1984] *Let*

i) *the distribution function F be twice differentiable on its support (a,b), where $a := \sup\{x : F(x) = 0\}$ and $b := \inf\{x : F(x) = 1\}$,*

ii) *the density function $f := F'$ be positive on (a,b),*

iii) *the bound*

$$\sup_{0<t<1} t(1-t)\frac{|f' \circ F^{-1}(t)|}{\{f \circ F^{-1}(t)\}^2} \leq \gamma$$

hold true for some finite $\gamma > 0$.

Then there exists a finite constant $\mathsf{C} < \infty$ such that the following two statements

$$\limsup_{n\to\infty} n\{\log\log n\}^{-(1+\gamma)} \sup_{[1/(n+1), n/(n+1)]} |f \circ F^{-1}\gamma_n - \gamma_n^U| \leq \mathsf{C} \quad \text{a.s.}$$

$$\limsup_{n\to\infty} n\{\log\log n\}^{-\gamma}\{\log n\}^{-(1+\epsilon)(\gamma-1)} \sup_{[1/(n+1), n/(n+1)]} |f \circ F^{-1}\gamma_n - \gamma_n^U| \leq \mathsf{C} \quad \text{a.s.},$$

hold true provided that, respectively, $\gamma \leq 1$ and $\gamma > 1$, where $\epsilon > 0$ is any fixed number.

In the following theorem the rates of convergence of the processes $f \circ F^{-1}\gamma_n - \gamma_n^U$ are considered uniformly over the interval $(0,1)$.

Throughout the paper we use $\|\ \|$ to denote the sup-functional $\sup\{|\cdot| : t \in (0,1)\}$.

Theorem 1.2 [M. Csörgő and Révész, 1978] *In addition to conditions* i), ii) and iii) *of Theorem* 1.1 *we also assume that*

iv) *the two limits $A := \lim_{x\downarrow a} f(x)$ and $B := \lim_{x\uparrow b} f(x)$ are finite.*

If

iv-a) *both limits A and B are positive,*

then there exists a finite constant $\mathsf{C} < \infty$ *such that*

$$\limsup_{n\to\infty} n\{\log\log n\}^{-1}\|f\circ F^{-1}\gamma_n - \gamma_n^U\| \le \mathsf{C} \quad \text{a.s.}$$

If, however,

iv-b) *the limit $A = 0$ (resp., $B = 0$), then we assume also that the density function f is non-decreasing on a right-neighbourhood of a (resp., non-increasing on a left-neighbourhood of b).*

Then there exists a finite constant $\mathsf{C} < \infty$ *such that the following three statements*

$$\limsup_{n\to\infty} n\{\log\log n\}^{-1}\|f\circ F^{-1}\gamma_n - \gamma_n^U\| \le \mathsf{C} \quad \text{a.s.},$$

$$\limsup_{n\to\infty} n\{\log\log n\}^{-2}\|f\circ F^{-1}\gamma_n - \gamma_n^U\| \le \mathsf{C} \quad \text{a.s.},$$

$$\limsup_{n\to\infty} n\{\log\log n\}^{-\gamma}\{\log n\}^{-(1+\epsilon)(\gamma-1)}\|f\circ F^{-1}\gamma_n - \gamma_n^U\| \le \mathsf{C} \quad \text{a.s.}$$

hold true provided that, respectively, $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$, where $\epsilon > 0$ is any fixed number.

Our reason for spelling out the conclusions of Theorem 1.1 under the conditions i), ii) and iii) is to call attention to the fact that under these conditions one can already accommodate all the jumps of the processes γ_n and γ_n^U, and thus successfully relate all the empirical quantiles of γ_n to those of its uniform version γ_n^U over the sequences of expanding intervals $[1/(n+1), n/(n+1)]$. The additional conditions on the density function f as in iv-a) and iv-b) of Theorem 1.2 are needed only for the sake of further smoothing the "theoretical tails" near the end-points a and b of the support of F, which in turn help us to conclude the following Corollaries 1.1 and 1.2.

Corollary 1.1 [M. Csörgő and Révész, 1978] *Under the conditions of Theorem* 1.2*, the statement*

$$\sqrt{\frac{n}{2\log\log n}} f\circ F^{-1}\gamma_n \rightsquigarrow \mathcal{H} \quad a.s. \tag{1.5}$$

holds true with respect to $\|\cdot\|$ on $D[0,1]$, where $\mathcal{H}$ is the Finkelstein (1971) *class of all absolutely continuous functions h on $[0,1]$ such that $h(0) = 0 = h(1)$ and $\int_0^1 \{h'(s)\}^2 ds \le 1$.*

The notation $\rightsquigarrow$ used above in (1.5) stands for saying that for almost all elementary events ω the set $\Gamma := \{\sqrt{n/(2\log\log n)} f\circ F^{-1}\gamma_n(\cdot,\omega) : n \in \mathbf{N}\}$ is relatively compact in the space $D[0,1]$ equipped with the sup-norm $\|\cdot\|$, and the set of all limit points of Γ is $\mathcal{H}$.

Corollary 1.2 [M. Csörgő and Révész, 1978] *Under the conditions of Theorem* 1.2*, the statement*

$$\sqrt{n} f\circ F^{-1}\gamma_n \to_d \mathcal{B} \tag{1.6}$$

holds true in the space $D[0,1]$ endowed with the Skorohod J_1 topology, where $\mathcal{B}$ is a Brownian bridge on $[0,1]$.

Statements (1.5) and (1.6) constitute two basic results upon which estimation of the quantile function F^{-1} and its various functionals of interest are usually built on (cf. Section 3 below for more details on the subject). The importance of quantiles in the statistical sciences has been discussed by Bahadur (1966), Bickel (1967), Chernoff, Gastwirth and Jones (1967), Kiefer (1967, 1970), Shorack (1972a,b), Parzen (1979a,b,c, 1980), M. Csörgő and Révész (1978, 1981), Davis and Steinberg (1986), M. Csörgő (1983, 1986), Eubank (1986), Heyde (1986), Shorack and Wellner (1986), M. Csörgő, S. Csörgő and Horváth [CsCsH](1986), M. Csörgő and Horváth (1993), and Zitikis (1998), among many others. Naturally, establishing as weak as possible assumptions under which statements (1.5) and (1.6) continue to hold true is of interest and practical importance. Therefore, in order to understand the global role of the full set of conditions on F in both Corollaries 1.1 and 1.2, as well as to find out to what extent these conditions can be relaxed, we shall now outline proofs of these two corollaries.

The proofs of both corollaries start with the following fundamental result of Kiefer (1970): The statement

$$\limsup_{n\to\infty} n^{3/4}\{\log\log n\}^{-1/4}\{\log n\}^{-1/2}\|R_n^U\| = 2^{-1/4} \quad a.s. \tag{1.7}$$

holds true, where

$$R_n^U := \gamma_n^U + \beta_n^U$$

denotes the $(0,1)$-uniform Bahadur-Kiefer process. (We also refer to Shorack, 1982, for a short and insightful proof of statement (1.7).) Consequently, under the assumptions of Theorem 1.2 we have (cf. M. Csörgő and Révész, 1978) that the general Bahadur-Kiefer process

$$R_n := f\circ F^{-1}\gamma_n + \beta_n^U$$

is such that

$$\limsup_{n\to\infty} n^{3/4}\{\log\log n\}^{-1/4}\{\log n\}^{-1/2}\|R_n\| = 2^{-1/4} \quad a.s. \tag{1.8}$$

If we now combine statement (1.8) with the following Finkelstein's (1971) law of the iterated logarithm [LIL]

$$\sqrt{\frac{n}{2\log\log n}}\beta_n^U \rightsquigarrow \mathcal{H} \quad a.s. \tag{1.9}$$

that holds true with respect to $\|\ \|$ in $D[0,1]$, then we shall obtain Corollary 1.1. On the other hand, if we combine statement (1.8) with the following Donsker's (1952) result

$$\sqrt{n}\beta_n^U \to_d \mathcal{B} \tag{1.10}$$

that holds true in the space $D[0,1]$ endowed with the Skorohod J_1 topology, then we shall get Corollary 1.2.

It is obvious from the above discussion that the respective proofs of statements (1.5) and (1.6) do not need such fast rates of convergence of $\|f\circ F^{-1}\gamma_n - \gamma_n^U\|$ to 0 as those given in Theorem 1.2, nor even those of statement (1.8) either. We though note in passing that the latter statement is of independent interest and proving it under the assumptions of Theorem 1.2 was one of the main motivations of M. Csörgő and Révész (1978). As to concluding statements (1.5) and (1.6), it is easy to see that one only needs to establish the following two statements:

$$n^{1/2}\{\log\log n\}^{-1/2}\|f\circ F^{-1}\gamma_n - \gamma_n^U\| \to_{a.s.} 0, \tag{1.11}$$

$$n^{1/2}\|f\circ F^{-1}\gamma_n - \gamma_n^U\| \to_P 0, \tag{1.12}$$

when $n \to \infty$. Since the respective rates of convergence to 0 of $\|f \circ F^{-1}\gamma_n - \gamma_n^U\|$ in both statements (1.11) and (1.12) are considerably slower than those stated in Theorem 1.2, one can therefore expect to have both statements (1.11) and (1.12) under weaker assumptions on F than those inherited from Theorem 1.1 for proving them. We are now to demonstrate this fact in our Theorem 1.3 below.

Given two functions g_1 and g_2, we use the notation $g_1(t) \asymp g_2(t)$, $t \to t_0$, if both statements $\limsup_{t\to t_0} |g_1(t)/g_2(t)| < \infty$ and $\limsup_{t\to t_0} |g_2(t)/g_1(t)| < \infty$ hold true.

Assumption 1.1 Let the distribution function F be absolutely continuous, and let the density-quantile function $f \circ F^{-1}$ be

v) continuous on the open interval $(0,1)$,

vi) (strictly) positive on the open interval $(0,1)$,

vii) such that the relations

$$f \circ F^{-1}(t) \asymp t^{\tau_1} S_1(\frac{1}{t}), \quad t \downarrow 0, \tag{1.13}$$

$$f \circ F^{-1}(t) \asymp (1-t)^{\tau_2} S_2(\frac{1}{1-t}), \quad t \uparrow 1, \tag{1.14}$$

hold true for some numbers $\tau_1, \tau_2 \geq 0$ and some slowly varying functions S_1, S_2. If $\tau_1 = 0$ (resp., $\tau_2 = 0$), then we assume $S_1(\frac{1}{t}) \asymp 1$ when $t \downarrow 0$ (resp., $S_2(\frac{1}{1-t}) \asymp 1$ when $t \uparrow 1$).

If it is not stated explicitely otherwise, Assumption 1.1 is assumed throughout the paper.

Assumption 1.1 was inspired by results and discussions in Shorack (1972a, b), M. Csörgő and Révész (1978), Parzen (1979a,b,c, 1980), CsCsHR (1984), M. Csörgő and Horváth (1993). It is interesting to note that Assumption 1.1 is essentially the same as assumption iii) of Theorem 1.1, though the advantage of Assumption 1.1 is that it does not require twice differentiability of F. Indeed, since the value of $\gamma > 0$ of assumption iii) becomes irrelevant when deducing the validity of statements (1.11) and (1.12) via Theorem 1.1, assumption iii) in this context therefore reduces to the following one

$$\sup_{0<t<1} t(1-t)\frac{|f' \circ F^{-1}(t)|}{f \circ F^{-1}(t)^2} \quad \left[= \sup_{0<t<1} t(1-t)\frac{|(d/dt)f \circ F^{-1}(t)|}{f \circ F^{-1}(t)} \right] < \infty. \tag{1.15}$$

Furthermore, upon noticing that assumption (1.15) is, in fact, an assumption on the tails of the density-quantile function $f \circ F^{-1}$, it can be reformulated as the following two assumptions

$$\tau_1 := \limsup_{t\downarrow 0} t\frac{|(d/dt)f \circ F^{-1}(t)|}{f \circ F^{-1}(t)} < \infty, \tag{1.16}$$

$$\tau_2 := \limsup_{t\uparrow 1} (1-t)\frac{|(d/dt)f \circ F^{-1}(t)|}{f \circ F^{-1}(t)} < \infty. \tag{1.17}$$

Discussions and results in Parzen (1979a,b,c, 1980), Seneta (1976), and Bingham, Goldie and Teugels [BGT] (1987) show that assumptions (1.16) and (1.17) with F twice differentiable carry essentially the same message as Assumption 1.1 without twice differentiability of F.

Theorem 1.3 *Both statements* (1.11) *and* (1.12) *hold true. Equivalently, the following two statements*

$$n^{1/2}\{\log\log n\}^{-1/2}\|R_n\| \to_{a.s.} 0, \tag{1.18}$$

$$n^{1/2}\|R_n\| \to_P 0 \tag{1.19}$$

hold true when $n \to \infty$.

Statements (1.18) and (1.19) taken together with the above mentioned Finkelstein (1971) and Donsker (1952) results (cf. statements (1.9) and (1.10), respectively) immediately imply the following corollary.

Corollary 1.3 *Strassen's type LIL* (*cf. statement* (1.5)) *for, and convergence in distribution* (*cf. statement* (1.6)) *of, the general quantile process* $f \circ F^{-1}\gamma_n$ *hold true.*

Further to our discussion given just below Theorem 1.2 about the conditions of Theorem 1.1 versus the additional ones of Theorem 1.2, we note that for concluding only what we want now, namely, (1.18)-(1.19) and hence also Corollary 1.3, the conditions of Assumption 1.1 replace those of Theorems 1.1 and 1.2 at a stroke.

In addition to the strong limit theorem (1.7), Kiefer (1970) also proved the following fundamental statement

$$n^{3/4}(\log n)^{-1/2}\|R_n^U\| \to_d \sqrt{\|\mathcal{B}\|} \tag{1.20}$$

when $n \to \infty$. In fact, more generally, Kiefer (1970) proved

$$\lim_{n\to\infty} n^{3/4}(\log n)^{-1/2}\|R_n^U\|/\sqrt{\|\mathcal{B}\|} \to_P 1 \tag{1.21}$$

and announced also, without publishing his proof, that the latter result holds almost surely as well. The first published proof of this almost sure version was given by Deheuvels and Mason (1990). The latter paper also has a complete survey of the developments in the area.

In words, statement (1.20) says that the appropriately normalized sup-functional $\|\ \|$ of the $(0,1)$-uniform Bahadur-Kiefer process R_n^U converges in distribution to the non-degenerate random variable $\sqrt{\|\mathcal{B}\|}$. It is interesting to note, however, that there can be no convergence in distribution of the Bahadur-Kiefer process R_n^U itself. This fact follows for example from comparing results of Kiefer (1967,1970) on the pointwise and sup-norm behaviour of R_n^U. It was formally proved first by Vervaat (1972b), and we record it here for our convenience as the next theorem.

Theorem 1.4 [Vervaat, 1972b] *The statement*

$$a_n R_n^U \to_d Y \tag{1.22}$$

cannot hold true in the space $D[0,1]$ *for any sequence* $\{a_n\}$ *of positive real numbers and any non-degenerate random element* Y *of* $D[0,1]$.

In a very elegant way, Vervaat (1972b) based his proof of the latter result on the following theorem.

Theorem 1.5 [Vervaat, 1972a] *When multiplied by n, the process*

$$V_n^U(t) := \int_0^t R_n^U(s)ds, \quad 0 \le t \le 1,$$

converges in distribution to $\frac{1}{2}\mathcal{B}^2$, that is to say, the statement

$$2nV_n^U \to_d \mathcal{B}^2 \tag{1.23}$$

holds true in the space $C[0,1]$ endowed with the topology of uniform convergence.

Having statement (1.23), Vervaat (1972b) argues that if statement (1.22) were true, then, using the Type Convergence Theorem [TCT] (cf., for example, Section 8.5 in BGT (1987) for results and references on the subject), one would then be able to replace a_n in (1.22) by n and, on account of Vervaat's statement (1.23), one would then have the equality

$$\int_0^\bullet Y(s)ds =_d \frac{1}{2}\mathcal{B}^2. \tag{1.24}$$

But the Brownian bridge $\mathcal{B}$ is a.s. nowhere differentiable and, consequently, the relation (1.24) cannot be true. Therefore, statement (1.22) cannot be true either. We conclude this paragraph with noting that M. Csörgő and Shi (1998) recently proved that the rate of convergence in distribution of the L_p-norm of the Bahadur-Kiefer process differs from that of the sup-norm of the Bahadur-Kiefer process in such a way that both simultaneously cannot be implied by having the statement of (1.22) with any sequence $\{a_n\}$. The latter fact obviously implies yet another proof of Theorem 1.4.

When investigating Strassen's LIL for the so-called Lorenz process, M. Csörgő and Zitikis (1996a) introduced, and used in a decisive way, the following process

$$V_n(t) := \int_0^t \gamma_n(s)ds + \int_{-\infty}^{F^{-1}(t)} \beta_n(x)dx, \quad 0 < t < 1.$$

It is easy to check that if F is absolutely continuous, then the representation

$$V_n(t) = \int_0^t R_n(s)dF^{-1}(s)$$

holds true for all $0 \le t \le 1$. Therefore, it now becomes obvious that in the $(0,1)$-uniform case the process V_n equals to the process V_n^U that was introduced and investigated by Vervaat (1972a,b). Therefore, we called the process V_n the (general) Vervaat process and, consequently, V_n^U the $(0,1)$-uniform Vervaat process (cf. Zitikis, 1998, for more mathematical and historical details on the subject).

The law of the iterated logarithm for, and convergence in distribution of, the (general) Vervaat process V_n can easily be deduced from the following theorem concerning the process

$$\Upsilon_n := f \circ F^{-1} V_n - \frac{1}{2}\{\beta_n^U\}^2.$$

Theorem 1.6 *The following two statements*

$$\frac{n}{\log\log n} \, \|\Upsilon_n\| \to_{a.s.} 0, \tag{1.25}$$

$$n \, \|\Upsilon_n\| \to_P 0 \tag{1.26}$$

hold true when $n \to \infty$.

Even though (under more stringent assumptions on F) one can obtain faster rates of convergence to 0 of the sup-functional $\|\cdot\|$ of the process Υ_n, statements (1.25) and (1.26) are exactly the statements that are needed to deduce the following two corollaries concerning, respectively, Strassen's type LIL for, and convergence in distribution of, the (general) Vervaat process V_n.

Corollary 1.4 *We have*

$$\frac{n}{\log\log n} f\circ F^{-1} V_n \rightsquigarrow \mathcal{H}^2 \quad a.s.$$

with respect to $\|\cdot\|$ *on* $C[0,1]$, *where* $\mathcal{H}^2 := \{h^2 :\ h\in\mathcal{H}\}$.

Corollary 1.5 *We have*

$$2n f\circ F^{-1} V_n \to_d \mathcal{B}^2$$

in the space $C[0,1]$ *endowed with the topology of uniform convergence.*

In the $(0,1)$-uniform case Corollary 1.4 was first proved by Vervaat (1972a), and Corollary 1.5 in this case coincides with Theorem 1.5. In the general case but under stronger requirements than those of Assumption 1.1, statement (1.25) and Corollary 1.4 were proved in M. Csörgő and Zitikis (1996b).

To conclude this section we note that the process Υ_n plays a similar role in the asymptotic theory of the Vervaat process V_n as the Bahadur-Kiefer process R_n does in the asymptotic theory of the quantile process γ_n. Namely, both of them are used to convert "difficult" processes V_n and γ_n into "easy" processes $\frac{1}{2}\{\beta_n^U\}^2$ and β_n^U, respectively. Therefore, just like in the case of the Bahadur-Kiefer process R_n (cf., for example, Bahadur, 1966, Kiefer, 1967, 1970; M. Csörgő and Révész, 1978; CsCsHR, 1984; Deheuvels and Mason, 1990, Ralescu, 1992, 1995, 1996; Shi, 1996, 1997), it may be of independent interest to obtain exact rates of strong and weak convergence for the appropriately normalized L_p- and sup-functionals of the process Υ_n and also of the processes

$$\Upsilon_n^* := f\circ F^{-1}V_n + \frac{1}{2}\beta_n^U f\circ F^{-1}\gamma_n,$$
$$\Upsilon_n^{**} := f\circ F^{-1}V_n - \frac{1}{2}\{f\circ F^{-1}\gamma_n\}^2,$$

as well as for the pointwise asymptotic behaviour of these three processes. In the $(0,1)$-uniform case, all of these problems have already been solved. For example, the following theorem, which parallels the results of Kiefer (1967, 1970) concerning the Bahadur-Kiefer process R_n^U, gives a complete description of the pointwise behaviour of the $(0,1)$-uniform version Υ_n^U of the process Υ_n, and also of the corresponding $(0,1)$-uniform versions of the processes Υ_n^* and Υ_n^{**}.

Theorem 1.7 [Csáki, M. Csörgő, Földes, Shi, and Zitikis, 1999] *For every fixed* $t\in(0,1)$, *we have*

$$n^{5/4}\Upsilon_n^U(t) \to_d \frac{1}{3^{1/2}}(t(1-t))^{3/4}G_1(|G_2|)^{3/2}, \tag{1.27}$$

$$n^{5/4}|\Upsilon_n^U(t)| \to_d \frac{1}{3^{1/2}}(t(1-t))^{3/4}|G_1|(|G_2|)^{3/2}, \tag{1.28}$$

when $n\to\infty$, *where* G_1 *and* G_2 *are independent standard Gaussian random variables, and we also have*

$$\limsup_{n\to\infty}\frac{n^{5/4}}{(\log\log n)^{5/4}}|\Upsilon_n^U(t)| = (t(1-t))^{3/4}\frac{2^{7/4}3^{1/4}}{5^{5/4}} \quad \text{a.s.}. \tag{1.29}$$

As to complete descriptions of the asymptotic behaviour of the sup- and L_p-norms of the process Υ_n^U, we refer, respectively, to Csáki, M. Csörgő, Földes, Shi, and Zitikis (1999) and M. Csörgő and Zitikis (1999a). A survey of the just mentioned results can be found in M. Csörgő and Zitikis (1999b).

The rest of this paper is organized as follows. In our next Section 2 we formulate without proofs two a.s. bounds for the process R_n (cf. Theorems 2.1 and 2.2) and one a.s. bound for the process Υ_n (cf. Theorem 2.3). It is then demonstrated in Section 2 that these three bounds imply both Theorems 1.3 and 1.6, which are, by the way, the only results of Section 1 that remain to be proved. We also note at the outset that Theorems 2.1-2.3 are more general results than those one really needs to have in order to derive Theorems 1.3 and 1.6. A justification for this generality is given in Section 3, where we discuss various constructions of confidence bands for the quantile function F^{-1}. In Section 3 we also investigate weighted versions of some of the results of Section 1. Section 4 is devoted to proving Theorems 2.1-2.3.

2 Proofs of the results of Section 1 via further results of interest

The only results of Section 1 that require proofs are Theorems 1.3 and 1.6. We denote

$$\delta_n := n^{-1} \log\log n$$

and start with the following auxiliary theorem that, together with Theorems 2.2 and 2.3 of this section, will be proved in Section 4.

Theorem 2.1 *For any $\delta > 0$ and $\epsilon > 0$, and for some $\epsilon_1 > 0$, the following bound*

$$|R_n(t)| \le ct^{\frac{1}{2}-\epsilon}(1-t)^{\frac{1}{2}-\epsilon}\Pi_n(\delta,\epsilon,\epsilon_1) + ct^{(\tau_1\wedge\frac{1}{2}-\epsilon)\vee 0}(1-t)^{(\tau_2\wedge\frac{1}{2}-\epsilon)\vee 0}\delta_n^{\frac{1}{2}+\epsilon_1}\{1+o_{a.s.}(1)\} \tag{2.1}$$

holds true for all $t \in (0,1)$, where

- *the constant c does not depend on δ, n and t,*
- *$o_{a.s.}(1)$ stands for a random variable which does not depend on t, but may depend on δ, ϵ and ϵ_1, and which converges to 0 almost surely when $n \to \infty$,*
- *the quantity $\Pi_n(\delta,\epsilon,\epsilon_1)$ on the right-hand side of (2.1) is defined as follows*

$$\Pi_n(\delta,\epsilon,\epsilon_1) := \|\beta_n^U\|\ o_{a.s.}(1) + c_\delta\|R_n^U\| + \delta_n^{\frac{1}{2}+\epsilon_1}\{1+o_{a.s.}(1)\}$$
$$+\,\delta^\epsilon\left(\sup_{t\in[25\delta_n,1-25\delta_n]}\left|\frac{\gamma_n^U(t)}{\sqrt{t(1-t)}}\right| + \sup_{t\in[25\delta_n,1-25\delta_n]}\left|\frac{\beta_n^U(t)}{\sqrt{t(1-t)}}\right|\right)\{1+o_{a.s.}(1)\},$$

where the constant c_δ may depend on δ but not on n and t.

If we bound t and $1-t$ by 1 on the right-hand side of (2.1), then we obtain the following corollary.

Corollary 2.1 *For any $\delta > 0$ and $\epsilon > 0$, and for some $\epsilon_1 > 0$, the following bound*

$$\|R_n\| \le c\Pi_n(\delta,\epsilon,\epsilon_1) \tag{2.2}$$

holds true for a constant c which does not depend on δ and n.

Corollary 2.1 is sufficient to deduce both statements (1.18) and (1.19). We are now to demonstrate this fact.

Proof of Theorem 1.3. Using the bound

$$\limsup_{n\to\infty} \delta_n^{-1/2}\left(\sup_{t\in[25\delta_n,1-25\delta_n]} \left| \frac{\gamma_n^U(t)}{\sqrt{t(1-t)}} \right| + \sup_{t\in[25\delta_n,1-25\delta_n]} \left| \frac{\beta_n^U(t)}{\sqrt{t(1-t)}} \right| \right) \le c \quad a.s. \tag{2.3}$$

(cf. Csáki, 1977, M. Csörgő and Révész, 1978, for more precise results), we obtain that

$$\limsup_{n\to\infty} \delta_n^{-1/2} \Pi_n(\delta,\epsilon,\epsilon_1) \le c\delta^{\epsilon} \quad a.s.$$

Letting $\delta > 0$ converge to 0 through rational points, we immediately get from bound (2.2) that the statement $\limsup \delta_n^{-1/2}\|R_n\| = 0$ a.s. holds true. Statements (1.18) and (1.11) are therefore proved.

In an analogous way but now using the following statement (cf. M. Csörgő and Horváth, 1993, for details, proofs and references)

$$n^{1/2}\left(\sup_{t\in[25\delta_n,1-25\delta_n]} \left| \frac{\gamma_n^U(t)}{\sqrt{t(1-t)}} \right| + \sup_{t\in[25\delta_n,1-25\delta_n]} \left| \frac{\beta_n^U(t)}{\sqrt{t(1-t)}} \right| \right) = \mathcal{O}_P(1) \tag{2.4}$$

instead of (2.3) , we arrive, via Corollary 2.1 again, at statements (1.19) and (1.12). This concludes the proof of Theorem 1.3. □

We note in passing that the constant 25 in the above statements (2.3) and (2.4) can be replaced by smaller ones (cf., for example, Shorack and Wellner, 1986, Einmahl and Mason, 1988, M. Csörgő and Horváth, 1993). However, an employment of such improvements in the present paper does not seem to be useful for our current needs.

As we have just seen, bound (2.1) of Corollary 2.1 is sufficient for the sake of proving Theorem 1.3. In Theorem 2.1 we have a more general result than that of Corollary 2.1 that will be useful in Section 3, where we discuss the important problem of constructing confidence bands for the quantile function F^{-1}. In Section 3 we shall also use, in a decisive way, the following theorem, which in the context of the current section can be considered as a supplement to Theorem 2.1.

Theorem 2.2 *For any $\delta > 0$ and $\epsilon > 0$, and some $\epsilon_1 > 0$, the following bound*

$$|R_n(t)| \le ct^{\frac{1}{2}-\epsilon}(1-t)^{\frac{1}{2}-\epsilon}\Pi_n(\delta,\epsilon,\epsilon_1), \tag{2.5}$$

holds true for all $t \in [1/(n+1), n/(n+1)]$, where the constant c does not depend on δ, n and t.

If we compare bounds (2.2) and (2.5), we shall notice that the restriction of t values to the interval $[1/(n+1), n/(n+1)]$ as in Theorem 2.2 has removed the asymptotically dominant second summand from the right-hand side of (2.1). This fact will play a decisive role in Section 3.

We are now to discuss Theorem 1.6 and its proof. To start with we formulate the following theorem whose proof will be given in Section 4.

Theorem 2.3 *For any $\delta > 0$ and $\epsilon > 0$, and some $\epsilon_1 > 0$, the following bound*

$$|\Upsilon_n(t)| \le ct^{1-\epsilon}(1-t)^{1-\epsilon}\tilde{\Pi}_n(\delta,\epsilon,\epsilon_1) \tag{2.6}$$

holds true for all $t \in (0,1)$, and for a constant c that does not depend on δ, n and t. The quantity $\tilde{\Pi}_n(\delta,\epsilon,\epsilon_1)$ on the right-hand side of (2.6) is defined as follows

$$
\begin{aligned}
\tilde{\Pi}_n(\delta,\epsilon,\epsilon_1) :=& c_\delta(\{\|R_n\| + \|R_n^U\|\}\|\beta_n^U\| + \{\|R_n\| + \|\beta_n^U\|\}\delta_n^{1/2+\epsilon_1} + \|\beta_n^U\|^2 o_{a.s.}(1)) \\
&+ \delta^{\epsilon_1}\Bigg(\sup_{t\in[25\delta_n,1-25\delta_n]} \left|\frac{\gamma_n^U(t)}{\sqrt{t(1-t)}}\right| \sup_{t\in[25\delta_n,1-25\delta_n]} \left|\frac{\beta_n^U(t)}{\sqrt{t(1-t)}}\right| \\
&\qquad + \sup_{t\in[25\delta_n,1-25\delta_n]} \left|\frac{\beta_n^U(t)}{\sqrt{t(1-t)}}\right|^2 \Bigg)\{1+o_{a.s.}(1)\} \\
&+ \delta_n^{\epsilon_1}\Bigg(\delta_n^{1/2} \sup_{t\in(0,1)} \left|\frac{\beta_n^U(t)}{t^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}}\right| \\
&\qquad + \sup_{t\in(0,1)} \left|\frac{\beta_n^U(t)}{t^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}}\right|^2\Bigg)\{1+o_{a.s.}(1)\} \\
&+ \delta_n^{1+\epsilon_1} o_{a.s.}(1),
\end{aligned}
$$

where the constant $c_\delta < \infty$ may depend on δ but not on n and t.

Proof of Theorem 1.6. It is easy to notice that in order to prove Theorem 1.6, one does not really need to have bound (2.6). Indeed, for the sake of proving Theorem 1.6 it is enough to have the following weaker bound

$$\|\Upsilon_n\| \le \tilde{\Pi}_n(\delta,\epsilon,\epsilon_1). \tag{2.7}$$

Then statement (1.25) of Theorem 1.6 follows from (2.7) via using statements (1.18), (1.9), together with the following two bounds

$$\limsup_{n\to\infty} \delta_n^{-1/2}\left(\sup_{t\in[25\delta_n,1-25\delta_n]} \left|\frac{\gamma_n^U(t)}{\sqrt{t(1-t)}}\right| + \sup_{t\in[25\delta_n,1-25\delta_n]} \left|\frac{\beta_n^U(t)}{\sqrt{t(1-t)}}\right|\right) \le c \quad a.s., \tag{2.8}$$

$$\limsup_{n\to\infty} \delta_n^{-1/2} \sup_{t\in(0,1)} \left|\frac{\beta_n^U(t)}{t^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}}\right| \le c \quad a.s. \tag{2.9}$$

(cf. Theorem 2 of M. Csörgő and Révész, 1978, and Theorem 3.2 on p.157 of of Csáki, 1977, concerning (2.8), and Corollary 2 of James, 1975, concerning (2.9)). Statement (1.26) of Theorem 1.6 follows from (2.7) via using statements (1.19), 1.10), (2.4), and

$$n^{1/2} \sup_{t\in(0,1)} \left|\frac{\beta_n^U(t)}{t^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}}\right| = \mathcal{O}_P(1) \tag{2.10}$$

(cf. CsCsHM, 1986, and M. Csörgő and Horváth, 1993, for proofs and further details concerning (2.10)). □

In Theorem 1.6 we obtained a sharper than necessary bound. This was done in order to describe a class of weight functions q that can be used to establish weighted versions of the process Υ_n and also have weighted versions of statements of Theorem 1.6 hold true. Specifically, from bound (2.6) we immediately deduce the following two corollaries.

Corollary 2.2 *If the weight function q satisfies the condition*

$$q(t) \geq ct^{1-\epsilon}(1-t)^{1-\epsilon}, \quad 0 < t < 1, \tag{2.11}$$

for some $c > 0$ and $\epsilon > 0$, then the following two statements

$$\frac{n}{\log\log n} \, \|\Upsilon_n/q\| \to_{a.s.} 0, \tag{2.12}$$

$$n \, \|\Upsilon_n/q\| \to_P 0 \tag{2.13}$$

hold true when $n \to \infty$.

In turn, statements (2.12) and (2.13) imply the following corollary.

Corollary 2.3 *If the weight function q is continuous on $(0,1)$, non-decreasing on a right-neighbourhood of 0, non-increasing on a left-neighbourhood of 1, and satisfies (2.11), then the following Strassen's type LIL*

$$\frac{n}{\log\log n} f \circ F^{-1} V_n/q \rightsquigarrow \mathcal{H}^2(q) := \{h^2/q : h \in \mathcal{H}\} \quad a.s. \tag{2.14}$$

holds true with respect to $\|\cdot\|$ on $C[0,1]$, and the convergence-in-distribution statement

$$2nf \circ F^{-1} V_n/q \to_d \mathcal{B}^2/q \tag{2.15}$$

holds true in the space $C[0,1]$ endowed with the topology of uniform convergence.

A precise description of the class of functions q such that statements (2.12)-(2.15) hold true still remains an open problem whose solution is definitely not within the scope of the present paper.

3 Asymptotic confidence bands for the quantile function F^{-1}

As we indicated in the previous section, Theorem 2.1 can be used to obtain weighted versions of both statements (1.18) and (1.19). The following theorem, which is an easy consequence of Theorem 2.1, is such a result.

Theorem 3.1 *If the weight function q is such that*

$$q(t) \geq ct^{(\tau_1 \wedge \frac{1}{2} - \epsilon)\vee 0}(1-t)^{(\tau_2 \wedge \frac{1}{2} - \epsilon)\vee 0}, \quad 0 < t < 1, \tag{3.1}$$

for some $c > 0$ and $\epsilon > 0$, then the following two statements

$$n^{1/2}\{\log\log n\}^{-1/2}\|R_n/q\| \to_{a.s.} 0, \tag{3.2}$$

$$n^{1/2}\|R_n/q\| \to_P 0 \tag{3.3}$$

hold true when $n \to \infty$.

Statements (3.2) and (3.3) easily imply the following theorem.

Theorem 3.2 *If the weight function q is continuous on $(0,1)$, non-decreasing on a right-neighbourhood of 0, non-increasing on a left-neighbourhood of 1, and satisfies assumption (3.1), then the following Strassen's type LIL*

$$\sqrt{\frac{n}{2\log\log n}} f\circ F^{-1}\gamma_n/q \rightsquigarrow \mathcal{H}(q) := \{h/q : h\in\mathcal{H}\} \quad a.s. \tag{3.4}$$

holds true with respect to $\|\cdot\|$ on $D[0,1]$, and the convergence-in-distribution statement

$$\sqrt{n} f\circ F^{-1}\gamma_n/q \to_d \mathcal{B}/q \tag{3.5}$$

holds true in $D[0,1]$ endowed with the Skorohod J_1 topology. The latter statement, in turn, implies

$$\sqrt{n}\sup_{t\in(0,1)} |f\circ F^{-1}(t)\gamma_n(t)/q(t)| \to_d \xi_q := \sup_{t\in(0,1)} |\mathcal{B}(t)/q(t)| \tag{3.6}$$

when $n\to\infty$.

Apart from its independent interest, statement (3.6) plays a decisive role in construction of confidence bands for the quantile function F^{-1}. Specifically, let $\alpha\in(0,1)$ be any fixed number, and let

$$z_{q,\alpha} := \inf\{z : \mathbf{P}\{\xi_q \le z\} \ge 1-\alpha\}.$$

With this notation, statement (3.6) immediately implies the following corollary.

Corollary 3.1 *If the weight function q is as in Theorem 3.2, then for any fixed $\alpha\in(0,1)$ the statement*

$$F^{-1}(t) \in \left[F_n^{-1}(t) - \frac{q(t)}{f\circ F^{-1}(t)}\frac{z_{q,\alpha}}{\sqrt{n}},\ F_n^{-1}(t) + \frac{q(t)}{f\circ F^{-1}(t)}\frac{z_{q,\alpha}}{\sqrt{n}}\right] \tag{3.7}$$

holds true over the interval $(0,1)$ with probability $1-\alpha+o(1)$ when $n\to\infty$.

The point z_α does not depend on any unknown parameter and can therefore be tabulated for any fixed weight function q. However, the confidence band in (3.7) is not readily applicable in practical situation due to the factor $f\circ F^{-1}(t)$ that needs to be estimated. Under additional assumptions on f, the latter problem was investigated by M. Csörgő and Révész (1984), Shorack and Wellner (1986), M. Csörgő and Horváth (1993).

To a certain degree, the estimation of, and thus additional assumptions on, the density-quantile function $f\circ F^{-1}$ can nevertheless be avoided by shifting, via the weight function q, the density quantile function $f\circ F^{-1}$ from the left-hand side to the right-hand side of (3.6) . Specifically, let the weight function q be q_0, where

$$q_0(t) := f\circ F^{-1}(t) t^{-\rho_1}(1-t)^{-\rho_2}$$

and the parameters ρ_1 and ρ_2 are such that

$$\rho_i \begin{cases} = 0 & \text{if } \tau_i = 0, \\ > 0 & \text{if } 0 < \tau_i \le \frac{1}{2}, \\ > \tau_i - \frac{1}{2} & \text{if } \tau_i > \frac{1}{2}. \end{cases} \tag{3.8}$$

It is easy to check that under Assumption 1.1 the weight function $q = q_0$ is such that assumption (3.1) holds true. Furthermore, it is also easy to check that $q_0 \ge q_1$, where q_1 is a non-decreasing on a right-neighbourhood of 0, non-increasing on a left-neighbourhood of 1, and is such that $1/q_1$ is square integrable over $(0,1)$. Consequently, using the last statement of Theorem 3.2, we conclude the following result.

Theorem 3.3 *If the parameters ρ_1 and ρ_2 are as in* (3.8), *then the statement*

$$\sqrt{n} \sup_{t\in(0,1)} |\gamma_n(t)t^{\rho_1}(1-t)^{\rho_2}| \to_d \xi_{\rho_1,\rho_2}(F) := \sup_{t\in(0,1)} |\mathcal{B}(t)t^{\rho_1}(1-t)^{\rho_2}/f\circ F^{-1}(t)| \tag{3.9}$$

holds true when $n\to\infty$.

Starting now with Theorem 3.3 instead of statement (3.6), we derive a somewhat different confidence band than that given in Corollary 3.1. Namely, let $\alpha\in(0,1)$ be fixed, and let

$$z_{\rho_1,\rho_2,\alpha}(F) := \inf\{z :\ \mathbf{P}\{\xi_{\rho_1,\rho_2}(F)\le z\}\ge 1-\alpha\}.$$

Then we have the following corollary.

Corollary 3.2 *If the parameters ρ_1 and ρ_2 are as in* (3.8), *then for any fixed $\alpha\in(0,1)$ the statement*

$$F^{-1}(t)\in\left[F_n^{-1}(t)-\frac{1}{t^{\rho_1}(1-t)^{\rho_2}}\frac{z_{\rho_1,\rho_2,\alpha}(F)}{\sqrt{n}},\ F_n^{-1}(t)+\frac{1}{t^{\rho_1}(1-t)^{\rho_2}}\frac{z_{\rho_1,\rho_2,\alpha}(F)}{\sqrt{n}}\right] \tag{3.10}$$

holds true over the interval $(0,1)$ with probability $1-\alpha+o(1)$ when $n\to\infty$.

Even though the (unknown) density-quantile function $f\circ F^{-1}$ has not been really removed from the confidence band in (3.10) due to the point $z_{\rho_1,\rho_2,\alpha}(F)$ still depending on $f\circ F^{-1}$, the estimation of the density-quantile function $f\circ F^{-1}$ can nevertheless be avoided now by using, for example, well-developed bootstrap methodologies (cf. M. Csörgő, S. Csörgő and Mason [CsCsM], 1984, CsCsH, 1986, S. Csörgő and Mason, 1989, M. Csörgő, Horváth and Kokoszka, 1998, for bootstaping empirical and related processes) via Theorem 3.3.

Naturally, the values of parameters ρ_1 and ρ_2 are crucial for constructing as narrow as possible confidence bands for F^{-1} in neighbourhoods of 0 and 1. Since assumption (3.8) implies that both ρ_1 and ρ_2 are non-negative, in most cases the confidence band given in (3.10) expands when $t\downarrow 0$ and $t\uparrow 0$. Unfortunately, this problem cannot be easily resolved since, as it easy to see, assumption (3.8) cannot be substantially relaxed without a modification of the original problem (3.3), or (3.5). Thus, assumption (3.8) cannot be substantially relaxed either.

In our next theorem we demonstrate that restricting the values t to the interval $[1/(n+1), n/(n+1)]$, we can relax assumption (3.8) significantly and in this way increase the range of values of ρ_1 and ρ_2.

Theorem 3.4 *If the parameters ρ_1 and ρ_2 are such that*

$$\rho_i > \tau_i - \frac{1}{2}, \tag{3.11}$$

then the statement

$$\sqrt{n} \sup_{t\in[1/(n+1),n/(n+1)]} |\gamma_n(t)t^{\rho_1}(1-t)^{\rho_2}| \to_d \xi_{\rho_1,\rho_2}(F) \tag{3.12}$$

holds true when $n\to\infty$, where $\xi_{\rho_1,\rho_2}(F)$ as in (3.9).

Similarly to the way we constructed the confidence band in (3.10), we now deduce from Theorem 3.4 the following corollary.

Corollary 3.3 *If the parameters* ρ_1 *and* ρ_2 *are as in* (3.11)*, then for any fixed* $\alpha \in (0,1)$ *the statement* (3.10) *holds true over the expanding interval* $[1/(n+1), n/(n+1)]$ *with probability* $1 - \alpha + o(1)$ *when* $n \to \infty$.

If we compare conditions (3.8) and (3.11), then we see that both of them coincide when $\tau_i > \frac{1}{2}$. However, condition (3.11) is weaker than (3.8) when $\tau_i \le \frac{1}{2}$.

We note in passing that the confidence bands for the quantile function F^{-1} in Doss and Gill (1992), that are appealing at the first sight, are obtained under the assumption that the density-quantile function $f \circ F^{-1}$ is continuous and (strictly) positive on the closed interval $[0,1]$. The latter assumption implies, in particular, that the distribution function F has a compact support.

More to the point along these lines, M. Csörgő and Horváth (1989) constructed confidence bands for F^{-1} even without the requirement of absolute continuity of the distribution function F, though, naturally, not over the closed interval $[0,1]$. As an illustration of their results we now formulate the following theorem.

Theorem 3.5 [M. Csörgő and Horváth, 1989] *Let* F *be continuous. Furthermore, let the weight function* q *be continuous on* $(0,1)$*, non-decreasing on a right-neighbourhood of* 0*, non-increasing on a left-neighbourhood of* 1*, and such that*

$$\int_0^1 \frac{1}{t(1-t)} \exp\Big\{ -\epsilon \frac{q^2(t)}{t(1-t)} \Big\} dt < \infty \tag{3.13}$$

for all $\epsilon > 0$*. Then for any fixed* $\alpha \in (0,1)$ *the statement*

$$F^{-1}(t) \in \Big[F_n^{-1}\big(t - q(t)\frac{z_\alpha}{\sqrt{n}}\big),\ F_n^{-1}\big(t + q(t)\frac{z_\alpha}{\sqrt{n}}\big) \Big], \quad \epsilon_n \le t \le 1 - \epsilon_n, \tag{3.14}$$

holds true with probability $1 - \alpha + o(1)$ *when* $n \to \infty$*, where* ϵ_n *is any sequence of positive numbers* $\epsilon_n \to 0$ *such that* $\sqrt{n}\epsilon_n \to \infty$.

We note in passing that the mere continuity of F in Theorem 3.5 was achieved by M. Csörgő and Horváth (1989) at the expense of restricting the values t to the interval $[\epsilon_n, 1-\epsilon_n]$, whereas the confidence bands of Corollaries 3.1-3.3 hold true either over the whole interval $(0,1)$ or over $[1/(n+1), n/(n+1)]$. All told, these facts reflect the complexity of the problem under investigation. To conclude this remark, we note that in M. Csörgő and Horváth (1989) the statement (3.14) was actually proved for the weight function $q(t) = 1$ only. However, in view of CsCsHM (1986), the incorporation of the weight function q satisfying (3.13) for all $\epsilon > 0$ into the statement (3.14) is an easy exercise.

The above discussion concerning confidence bands for F^{-1} has demonstrated the importance of statement (3.6) and that of its truncated version

$$\sqrt{n} \sup_{t \in [1/(n+1) \le t \le n/(n+1)]} |f \circ F^{-1} \gamma_n / q| \to_d \sup_{t \in (0,1)} |\mathcal{B}/q| \tag{3.15}$$

under various possible assumptions on q vis-á-vis the density quantile function $f \circ F^{-1}$. We now consider the validity of statement (3.15) in the indicated general form. The reason is that the interval $[1/(n+1), n/(n+1)]$, over which sup is taken on the left-hand side of (3.15), already includes all the empirical information that is carried by F_n^{-1} on F^{-1}.

We continue our discussion with the following result of Shorack (1972b).

Theorem 3.6 [Shorack, 1972b] *Let the weight function q be continuous on $(0,1)$, non-decreasing on a right-neighbourhood of 0, non-increasing on a left-neighbourhood of 1, and such that*

$$\int_0^1 \left\{ \frac{1}{q(t)} \right\}^2 dt < \infty. \tag{3.16}$$

Furthermore, let the density-quantile function $f \circ F^{-1}$ be continuous and positive on the open interval $(0,1)$, and let the bound

$$|f \circ F^{-1}(t)| R(t) \geq 1, \quad 0 < t < 1,$$

hold true for a non-increasing on a right-neighbourhood of 0, non-decreasing on a left-neighbourhood of 1 function R such that for each $\beta > 0$ in a right-neighbourhood of 0 there is a constant M_β such that the following two inequalities

$$R(\beta t) \leq M_\beta R(t), \tag{3.17}$$
$$R(1-\beta(1-t)) \leq M_\beta R(t) \tag{3.18}$$

hold true for all $t \in (0,1)$. Furthermore, assume that the statement

$$f \circ F^{-1}(t) R(t) \nu(t) \to 0 \tag{3.19}$$

holds true when $t \downarrow 0$ and $t \uparrow 1$ for a continuous function ν such that $\nu(0)$ and $\nu(1) = 0$ and such that

$$\int_0^1 \left\{ \frac{1}{\nu(t) q(t)} \right\}^2 dt < \infty. \tag{3.20}$$

Then statement (3.15) holds true.

We note in passing that condition (3.16) implies that of (3.13) (cf., for example, p. 27 of CsCsH, 1986). Hence, unlike in (3.14), however, we deal with taking sup over the interval $[1/(n+1), n/(n+1)]$. In this regard, conditions (3.19) and (3.20) imply, in particular, that the more exactly we are able describe the behaviour of the density-quantile function $f \circ F^{-1}(t)$ when $t \downarrow 0$ and $t \uparrow 1$, the larger class of weight functions q can be used in statement (3.15). In this sense the function ν defined in (3.19) and (3.20) can be interpreted as a penalty for the lack of information about the behaviour of the density-quantile function $f \circ F^{-1}(t)$ near 0 and 1.

Under more stringent assumptions on F than those given in Theorem 3.6 above, M. Csörgő (1986) and G.R. Shorack (cf. Theorem 2 on p. 642 in Shorack and Wellner, 1986) proved the following theorem.

Theorem 3.7 [M. Csörgő, 1986; G.R. Shorack, 1986] *Let assumptions* i)-iii) *of Theorem* 1.1 *hold true, and let the weight function q be as in Theorem* 3.5. *Then statement* (3.15) *holds true.*

Theorem 2.2 on p. 382 of M. Csörgő and Horváth (1993) demonstrates that assumptions i)-iii) [of Theorem 1.1] can be further weakened in Theorem 3.7. Namely, the following theorem holds true.

Theorem 3.8 [M. Csörgő and Horváth, 1993] *Let the density-quantile function $f \circ F^{-1}$ be continuous and positive on $(0,1)$, and let there exist two monotonous functions R_1 and R_1 satisfying the following two sets of conditions:*

$$\begin{cases} |f \circ F^{-1}(t)| R_1(t) \geq 1, \\ |f \circ F^{-1}(t)| R_2(t) \geq 1 \end{cases} \tag{3.21}$$

for all $t \in (0,1)$, *and*

$$\begin{cases} \limsup_{t\to 0} |f\circ F^{-1}(t)| R_1(\beta t) < \infty, \\ \limsup_{t\to 1} |f\circ F^{-1}(t)| R_2(1-\beta(1-t)) < \infty \end{cases} \tag{3.22}$$

for all $0<\beta<\infty$. *Then statement* (3.15) *holds true for any weight function* q *as in Theorem* 3.5.

We note in passing that if the function R_1 satisfies condition (3.17) and R_2 satisfies (3.18), then conditions (3.21) and (3.22) essentially require the existence of monotonous functions that bound the density-quantile function $f\circ F^{-1}$ in neighbourhoods of 0 and 1. Therefore, if we know or assume that the density-quantile function $f\circ F^{-1}$ is monotonous in neighbourhoods of 0 and 1 (cf. assumption iv-b) of Theorem 1.2), then assumptions (3.21) and (3.22) are, of course, void. If the density-quantile function $f\circ F^{-1}$ is regularly varying (as in Assumption 1.1), then assumptions (3.21) and (3.22) are satisfied since according to, for example, Theorem 1.5.3 of BGT (1987), every regularly varying function has a monotone equivalent.

Theorem 3.1(v) on p. 386-387 of M. Csörgő and Horváth (1993) further relaxes the assumption that (3.13) holds true for all $\epsilon>0$ at the expense of strengthening, however, the requirements on F, just like in Theorem 3.7. Thus the next theorem is to be compared to both Theorems 3.7 and 3.8.

Theorem 3.9 [M. Csörgő and Horváth, 1993] *Let assumptions* i)-iii) *of Theorem* 1.1 *hold true. Furthermore, let the function* q *be continuous on* $(0,1)$, *non-decreasing on a right-neighbourhood of* 0, *non-increasing on a left-neighbourhood of* 1, *and such that the assumption* (3.13) *is satisfied for some* $\epsilon>0$. *Then statement* (3.15) *holds true.*

The requirement that assumption (3.13) in Theorem 3.9 holds true for some $\epsilon>0$ cannot be relaxed, since this requirement is necessary and sufficient for the existence of the limiting non-degenerate random variable $\sup_{t\in(0,1)}|\mathcal{B}/q|$ (cf. CsCsHM, 1986, Shorack and Wellner, 1986, M. Csörgő, Shao and Szyszkowicz [CsShSz], 1991, M. Csörgő and Horváth, 1993).

We are now to give a few remarks concerning the so-called first type Vervaat problem that aims at describing the asymptotic behaviour of inverted processes via the known or assumed asymptotic behaviour of the original processes. (The second type Vervaat problem aims at describing the asymptotic behaviour of integrals of inverted processes such as, for example, the process V_n investigated in previous sections; cf. Zitikis, 1998, for a survey on the subject.) In great detail and with numerous examples and applications in diverse fields of mathematics and the mathematical sciences, the first type Vervaat problem was initiated and investigated by Vervaat (1972a,b). Whitt (1980) made further far-reaching contributions in the area. Ralescu and Puri (1996) obtained, to the best of our knowledge, the most complete and general solution of the first type Vervaat problem, and accompanied their solution with a number of examples of importance in diverse areas of Probability Theory and Mathematical Statistics.

Since inverted processes are at least as important in statistical sciences as non-inverted ones, there has been an increasing interest in obtaining as general solutions of the first Vervaat problem as possible. The work by Doss and Gill (1992) is sometimes considered as one of the most important contributions in the area. While it is succinct in their context, we note that Doss and Gill (1992) base their considerations in general on their result which says, in particular, that if the density-quantile function $f\circ F^{-1}$ is

- continuous on the closed interval $[0,1]$,
- (strictly) positive on the closed interval $[0,1]$,

then the statement

$$\sqrt{n}\{F_n^{-1} - F^{-1}\} \to_d -\frac{\mathcal{B}}{f \circ F^{-1}} \tag{3.23}$$

holds true. However, as we already indicated above, these assumptions that clearly yield statement (3.23) are very restrictive for general use, since they exclude all distribution functions with non-compact supports.

To conclude this section we note that, to the best of our knowledge, it was Shorack (1972b) (cf., for example, Theorem 3.6 above) who gave the first and remarkably general solution of the first Vervaat problem, as we call the problem nowadays. Theorems 3.1-3.5 and 3.7-3.9 are examples of further developments in the area and give a fairly exhaustive solution to the problem in hand.

4 Proofs of the theorems of Section 2

In this section we prove a number of lemmas from which Theorems 2.1-2.3 follow. In particular, Lemmas 4.1-4.3 imply Theorem 2.2. Lemma 4.4 together with Theorem 2.2 imply Theorem 2.1. Finally, Lemmas 4.5-4.8 imply Theorem 2.3.

We assume throughout this section that Assumption 1.1 holds true. We also recall that $o_{a.s.}(1)$ stands for random variables which do not depend on t and converge to 0 almost surely when $n \to \infty$ (cf. the formulation of Theorem 2.1 for complete details).

Lemma 4.1 *Bound* (2.5) *holds true for all* $t \in [\delta, 1-\delta]$*, where* $\delta > 0$ *is an arbitrary but fixed number.*

Proof. Since $F_n^{-1} = F^{-1} \circ E_n^{-1}$ and $(d/ds)F^{-1}(t) = 1/f \circ F^{-1}(t)$, the Taylor formula implies the representation

$$f \circ F^{-1}(t)\gamma_n(t) - \gamma_n^U(t) = \gamma_n^U(t) \int_0^1 \left\{ \frac{f \circ F^{-1}(t)}{f \circ F^{-1}(\xi_s)} - 1 \right\} ds, \tag{4.1}$$

where $\xi_s := t + s\gamma_n^U(t)$. Since $\|\gamma_n^U\|$ converges to 0 almost surely when $n \to \infty$, and the density-quantile function $f \circ F^{-1}$ is uniformly continuous on $[\delta, 1-\delta]$ by Assumption 1.1, we obtain from representation (4.1) that the bound

$$\begin{aligned} |f \circ F^{-1}(t)\gamma_n(t) - \gamma_n^U(t)| &\le t^A(1-t)^A \|\gamma_n^U\| \; o_{a.s.}(1) \\ &= t^A(1-t)^A \|\beta_n^U\| \; o_{a.s.}(1) \end{aligned}$$

holds true for any fixed $A > 0$. Consequently, we have the bound

$$|R_n(t)| \le t^A(1-t)^A \{\|\beta_n^U\| \; o_{a.s.}(1) + c_\delta \|R_n^U\|\}$$

that completes the proof of Lemma 4.1. □

Lemma 4.2 *Bound* (2.5) *holds true for all* $t \in [25\delta_n, \delta] \cup [1-\delta, 1-25\delta_n]$*, where* $\delta > 0$ *is a sufficiently small but fixed number.*

Proof. Let $t \in [25\delta_n, \delta]$. The Taylor formula (cf. representation (4.1)) implies the bound

$$f \circ F^{-1}(t)|\gamma_n(t)| \le |\gamma_n^U(t)| \int_0^1 \frac{f \circ F^{-1}(t)}{f \circ F^{-1}(\xi_s)} ds, \tag{4.2}$$

where $\xi_s := t + s\gamma_n^U(t)$. Since $\delta > 0$ is sufficiently small by assumption, we use Assumption 1.1 in order to replace both $f \circ F^{-1}(t)$ and $f \circ F^{-1}(\xi_s)$ on the right hand-side of (4.2) by $t^{\tau_1} S_1(1/t)$ and $\xi_s^{\tau_1} S_1(1/\xi_s)$, respectively. Consequently, the bound

$$f \circ F^{-1}(t)|\gamma_n(t)| \le c|\gamma_n^U(t)| \int_0^1 \left\{ \frac{t^{\tau_1} S_1(1/t)}{\xi_s^{\tau_1} S_1(1/\xi_s)} \right\} ds \{1 + o_{a.s.}(1)\} \tag{4.3}$$

holds true for a finite constant c that does not depend on δ, n, and t. It is shown on p.889 of M. Csörgő and Révész (1978) that t/ξ_s does not exceed $5 + o_{a.s.}(1)$ (≤ 10 for large n). In a similar way one can check that t/ξ_s is not smaller than $5/9 - o_{a.s.}(1)$ ($\ge 1/10$ for large n). Applying these two facts on the right-hand side of (4.3), we get the bound

$$f \circ F^{-1}(t)|\gamma_n(t)| \le c|\gamma_n^U(t)| \sup_{\lambda \in [1/10, 10]} \left\{ \frac{S_1(1/t)}{S_1(\lambda/t)} \right\} \{1 + o_{a.s.}(1)\}. \tag{4.4}$$

The Uniform Convergence Theorem [UCT] for slowly varying functions (cf., for example, Theorem 1.2.1 on p.6 of BGT, 1987) tells us that the supremum on the right-hand side of (4.4) does not exceed, say, 10 if t is sufficiently small (which is easily achieved by taking $\delta > 0$ sufficiently small and upon recalling that $0 \le t \le \delta$). In this way we arrive at the bound

$$f \circ F^{-1}(t)|\gamma_n(t)| \le c|\gamma_n^U(t)|\{1 + o_{a.s.}(1)\} \tag{4.5}$$

that holds true for all $t \in [25\delta_n, \delta]$. In a similar way one obtains the bound (4.5) for all $t \in [1-\delta, 1-25\delta_n]$. Consequently, for all $t \in [25\delta_n, \delta] \cup [1-\delta, 1-25\delta_n]$, the quantity $|R_n(t)|$ does not exceed

$$ct^{1/2}(1-t)^{1/2}\Bigg(\sup_{t\in[25\delta_n, 1-25\delta_n]} \left| \frac{\gamma_n^U(t)}{\sqrt{t(1-t)}} \right| + \sup_{t\in[25\delta_n, 1-25\delta_n]} \left| \frac{\beta_n^U(t)}{\sqrt{t(1-t)}} \right| \Bigg) \{1 + o_{a.s.}(1)\}.$$

The proof of Lemma 4.2 is now complete. □

Lemma 4.3 *Bound* (2.5) *holds true for all* $t \in [1/(n+1), 25\delta_n] \cup [1 - 25\delta_n, n/(n+1)]$.

Proof. Let $t \in [1/(n+1), 25\delta_n]$. The Taylor formula implies the representation

$$f \circ F^{-1}(t)\gamma_n(t) = \int_t^{E_n^{-1}(t)} \frac{f \circ F^{-1}(t)}{f \circ F^{-1}(s)} ds. \tag{4.6}$$

Due to Theorem 5(i) on p.80 in Wellner (1978), we have that, almost surely, the bounds $E_n^{-1}(t) \le E_n^{-1}(25\delta_n) \le c\delta_n$ hold true for sufficiently large n. Therefore, both t and $E_n^{-1}(t)$ can be made arbitrarily small, which enables us to use Assumption 1.1 on the right-hand side of (4.6) and obtain the bound

$$f \circ F^{-1}(t)|\gamma_n(t)| \le c\left| \int_t^{E_n^{-1}(t)} \frac{t^{\tau_1} S_1(1/t)}{s^{\tau_1} S_1(1/s)} ds \right| \{1 + o_{a.s.}(1)\}. \tag{4.7}$$

In order to estimate the ratio $S_1(1/t)/S_1(1/s)$ in (4.7), we use Potter's Theorem (cf. Theorem 1.5.6 on p.25 in BGT, 1987) and obtain that, for any $c > 1$ and $\rho > 0$, and for sufficiently large n (in order to make both t and $E_n^{-1}(t)$ sufficiently small), the bound

$$\frac{S_1(1/t)}{S_1(1/\xi_s)} \le c\left\{ \left(\frac{s}{t}\right)^\rho \vee \left(\frac{t}{s}\right)^\rho \right\} \tag{4.8}$$

holds true for all $t \in (0, 25\delta_n]$. We also observe that, due to Assumption 1.1, if we have $\tau_1 = 0$, then bound (4.8) holds true with $\rho = 0$. Consequently, bounds (4.7) and (4.8) imply the following one

$$f \circ F^{-1}(t)|\gamma_n(t)| \leq c\Delta(t)\{1 + o_{a.s.}(1)\}, \tag{4.9}$$

where we have denoted

$$\Delta(t) := \Big| \int_t^{E_n^{-1}(t)} \left(\frac{t}{s}\right)^{\tau_1} \left\{ \left(\frac{s}{t}\right)^{\rho} \vee \left(\frac{t}{s}\right)^{\rho} \right\} ds \Big|.$$

(We note in passing that ρ in the definition of $\Delta(t)$ is either 0 or positive depending on, respectively, whether $\tau_1 = 0$ or $\tau_1 > 0$.) If $E_n^{-1}(t) \geq t$, then $s \geq t$, and we therefore obtain the bound

$$\Delta(t) \leq c\{t + t^{\tau_1 - \rho} E_n^{-1}(t)^{1-(\tau_1 - \rho)}\}, \tag{4.10}$$

since we can always chose ρ so small that $\tau_1 - \rho \neq 1$. On the other hand, if $E_n^{-1}(t) < t$, then inequality (4.10) holds true with $\tau_1 - \rho$ replaced by $\tau_1 + \rho$, since ρ can always be chosen so small that $\tau_1 + \rho \neq 1$. Denote $\tau := \tau_1 \pm \rho$, where we have either $+$ or $-$ depending on, respectively, whether $E_n^{-1}(t) < t$ or $E_n^{-1}(t) \geq t$. Consequently, we have the bound

$$\Delta(t) \leq c\{t + t^{\tau} E_n^{-1}(t)^{1-\tau}\} \tag{4.11}$$

with $\tau = \tau_1 \pm \rho$. (It is obvious that τ can always be assumed to be non-negative since $\rho = 0$ if $\tau_1 = 0$ and $\rho > 0$ can be chosen arbitrarily small if $\tau_1 > 0$.) If $\tau \in [0, 1)$, then $1 - \tau > 0$. Therefore, using the bound $E_n^{-1}(t) \leq ct^{1-\lambda}\{1 + o_{a.s.}(1)\}$ that holds true for any fixed $\lambda > 0$ (cf. (8) on p.483 of Wellner, 1977), we get

$$\begin{aligned} \Delta(t) &\leq ct^{1-\lambda(1-\tau)}\{1 + o_{a.s.}(1)\} \\ &\leq ct^{1/2-\lambda(1-\tau)}\delta_n^{1/2}\{1 + o_{a.s.}(1)\}. \end{aligned} \tag{4.12}$$

If $\tau \in [1, \infty)$, then the bounds $t^{\tau} \leq t^{1/2}\delta_n^{\tau-1/2}$ and $E_n^{-1}(t)^{1-\tau} \leq U_{1:n}^{1-\tau}$ imply the bound $\Delta(t) \leq ct^{1/2}\delta_n^{1/2}(\delta_n/U_{1:n})^{\tau-1}$. Consequently, due to the fact that for any $\epsilon > 0$ the statement $\liminf_{n\to\infty} U_{1:n} n(\log n)^{1+\epsilon} = \infty$ holds true almost surely (cf. (3) on p.408 of Shorack and Wellner, 1986), we get the bound

$$\Delta(t) \leq ct^{1/2}\delta_n^{1/2}(\log n)^{(1+\epsilon)(\tau-1)} o_{a.s.}(1) \tag{4.13}$$

for any $\epsilon > 0$. Taking bounds (4.9), (4.12) and (4.13) together, we arrive at the following one

$$|f \circ F^{-1}(t)\gamma_n(t)| \leq ct^{1/2-\epsilon}\delta_n^{1/2+\epsilon_1}\{1 + o_{a.s.}(1)\} \tag{4.14}$$

that holds true for all $t \in [1/(n+1), 25\delta_n]$. In a similar way one arrives at bound (4.14) for all $t \in [1 - 25\delta_n, n/(n+1)]$, but with t on the right hand side of (4.14) replaced by $1 - t$. Consequently, the bound

$$|R_n(t)| \leq ct^{1/2-\epsilon}(1-t)^{1/2-\epsilon}\delta_n^{1/2+\epsilon_1}\{1 + o_{a.s.}(1)\} \tag{4.15}$$

holds true for all $t \in [1/(n+1), 25\delta_n] \cup [1 - 25\delta_n, n/(n+1)]$. This completes the proof of Lemma 4.3. □

Proof of Theorem 2.2 Theorem 2.2 is an elementary consequence of Lemmas 4.1-4.3. □

Lemma 4.4 *Bound*

$$|R_n(t)| \le ct^{(\tau_1\wedge\frac{1}{2}-\epsilon)\vee 0}(1-t)^{(\tau_2\wedge\frac{1}{2}-\epsilon)\vee 0}\delta_n^{\frac{1}{2}+\epsilon_1}\{1+o_{a.s.}(1)\}$$

holds true for all $t \in (0, 1/(n+1)] \cup [n/(n+1), 1)$.

Proof. We first estimate the quantity $|R_n(t)|$ for all $t \in (0, 1/(n+1)]$. Theorem 1 of Wellner (1977) tells us that $E_n(t) \le ct^{1-\lambda}\{1+o_{a.s.}(1)\}$ for any fixed $\lambda > 0$. Thus,

$$|\beta_n^U(t)| \le ct^{1-\lambda}\{1+o_{a.s.}(1)\} \tag{4.16}$$

$$\le ct^{1/2-\epsilon}\delta_n^{1/2+\epsilon}\{1+o_{a.s.}(1)\}. \tag{4.17}$$

We are now to estimate $|f\circ F^{-1}(t)\gamma_n(t)|$ for all $t \in (0, 1/(n+1)]$, Using inequalities (4.9) and (4.11) as well as the equality $E_n^{-1}(t) = U_{1:n}$ that holds true for all $t \in (0, 1/(n+1)]$, we obtain the bound

$$|f\circ F^{-1}(t)\gamma_n(t)| \le c\{t + t^{\tau}U_{1:n}^{1-\tau}\}. \tag{4.18}$$

Furthermore, since the following two statements

$$\limsup_{n\to\infty}\{n/\log n\}U_{1:n} < \infty \quad a.s.$$

(cf. Exercise 2(i) on p.408 of Shorack and Wellner, 1986) and, for sufficiently small $\epsilon_1 > 0$,

$$\limsup_{n\to\infty}\delta_n^{-1/2-\epsilon_1}\omega_n(c\delta_n^{1/2}) = 0 \quad a.s.$$

(cf., for example, Theorem 1 on p.542 of Shorack and Wellner, 1986) hold true, we easily obtain from (4.18) the following bound

$$|f\circ F^{-1}(t)\gamma_n(t)| \le ct^{(\tau_1\wedge\frac{1}{2}-\epsilon)\vee 0}\delta_n^{\frac{1}{2}+\epsilon_1}\{1+o_{a.s.}(1)\}. \tag{4.19}$$

The proof of bound (4.19) for all $t \in [1-1/n, 1)$ but with $(1-t)^{(\tau_2\wedge\frac{1}{2}-\epsilon)\vee 0}$ instead of $t^{(\tau_1\wedge\frac{1}{2}-\epsilon)\vee 0}$ is similar. This remark completes the proof of Lemma 4.4. □

Proof of Theorem 2.1. Theorem 2.1 follows immediately from Theorem 2.2 and Lemma 4.4. □

Lemma 4.5 *Bound* (2.6) *holds true for all* $t \in [\delta, 1-\delta]$, *where* $\delta > 0$ *is an arbitrary but fixed number.*

Proof. It is easy to check (cf. M. Csörgő and Zitikis, 1996b, and Zitikis, 1998) that the "asymptotic expansion"

$$V_n(t) = -\frac{1}{2}\beta_n^U(t)\gamma_n(t) + \rho_n(t), \quad 0 < t < 1, \tag{4.20}$$

holds true with the "remainder" term

$$\rho_n(t) := A_n(t) - \frac{1}{2}B_n(t) + \frac{1}{2}C_n(t), \quad 0 < t < 1,$$

where

$$A_n(t) := \int_{E_n^{-1}(t)}^{t}\{\beta_n^U(s) - \beta_n^U(t)\}dF^{-1}(s),$$

$$B_n(t) := \{\beta_n^U(t) + \gamma_n^U(t)\}\gamma_n(t) \quad [= R_n^U(t)\gamma_n(t)],$$

$$C_n(t) := \int_{E_n^{-1}(t)}^{t}\{F_n^{-1}(t) - 2F^{-1}(s) + F^{-1}(t)\}ds.$$

Using the "asymptotic expansion" of (4.20), we obtain the following representation for the process Υ_n:

$$\Upsilon_n(t) = -\frac{1}{2}R_n(t)\beta_n^U(t) + f\circ F^{-1}(t)\rho_n(t). \tag{4.21}$$

We are now to estimate both summands on the right-hand side of (4.21). The bound

$$|R_n(t)\beta_n^U(t)| \le c_\delta t^A(1-t)^A\|R_n\|\|\beta_n^U\| \tag{4.22}$$

is obvious and holds true for any fixed $A>0$. In order to estimate $f\circ F^{-1}(t)|\rho_n(t)|$, we start with the bound

$$f\circ F^{-1}(t)|A_n(t)| \le cf\circ F^{-1}(t)|\gamma_n(t)|\omega_n(|\gamma_n^U(t)|),$$

where

$$\omega_n(h) := \sup_{|u-v|\le h} |\beta_n^U(u)-\beta_n^U(v)|$$

is the oscillation moduli of the $(0,1)$-uniform empirical process β_n^U. Consequently, we have the bound

$$\begin{aligned} f\circ F^{-1}(t)|A_n(t)| &\le \{\|R_n\|+\|\beta_n^U\|\}\omega_n(\|\gamma_n^U\|) \\ &= \{\|R_n\|+\|\beta_n^U\|\}\omega_n(\|\beta_n^U\|), \end{aligned} \tag{4.23}$$

where the above equality holds true due to $\|\gamma_n^U\|=\|\beta_n^U\|$. Because of the two statements

$$\limsup_{n\to\infty} \delta_n^{-1/2}\|\beta_n\| \le c \quad a.s.$$

and, for sufficiently small $\epsilon_1>0$,

$$\limsup_{n\to\infty} \delta_n^{-1/2-\epsilon_1}\omega_n(c\delta_n^{1/2}) = 0 \quad a.s.$$

(cf., for example, Theorem 1 on p.542 of Shorack and Wellner, 1986), we get from (4.23) that

$$f\circ F^{-1}(t)|A_n(t)| \le t^A(1-t)^A\{\|R_n\|+\|\beta_n^U\|\}\delta_n^{\frac{1}{2}+\epsilon_1}o_{a.s.}(1). \tag{4.24}$$

The bounds

$$f\circ F^{-1}(t)|B_n(t)| \le ct^A(1-t)^A\{\|R_n\|+\|\beta_n^U\|\}\|R_n^U\| \tag{4.25}$$

$$\le ct^A(1-t)^A\{\|R_n\|+\|\beta_n^U\|\}\|\beta_n^U\| \tag{4.26}$$

are obvious. In order to estimate $f\circ F^{-1}(t)|C_n(t)|$ we proceed as follows. Changing the integration variable s in $C_n(t)$ by $t+s\gamma_n^U(t)$, and then applying the Taylor formula for F^{-1}, we obtain the representation

$$\begin{aligned} f\circ F^{-1}(t)C_n(t) &= -\{\gamma_n^U(t)\}^2 \\ &\times \int_0^1\int_0^1 \Big\{\frac{f\circ F^{-1}(t)}{f\circ F^{-1}(t+s_1\gamma_n^U(t))} - 2s\frac{f\circ F^{-1}(t)}{f\circ F^{-1}(t+s_1\gamma_n^U(t)s)}\Big\}ds_1ds. \end{aligned} \tag{4.27}$$

An elementary rearrangement of the right-hand side of (4.27) gives, in turn, the representation

$$\begin{aligned} f\circ F^{-1}(t)C_n(t) &= -\{\gamma_n^U(t)\}^2 \\ &\times \int_0^1\int_0^1 \Big\{\Big(\frac{f\circ F^{-1}(t)}{f\circ F^{-1}(t+s_1\gamma_n^U(t))}-1\Big) - 2s\Big(\frac{f\circ F^{-1}(t)}{f\circ F^{-1}(t+s_1\gamma_n^U(t)s)}-1\Big)\Big\}ds_1ds. \end{aligned} \tag{4.28}$$

Since $\|\gamma_n^U\|$ converges to 0 almost surely, and the density-quantile function $f \circ F^{-1}$ is uniformly continuous on every compact subinterval of $(0,1)$, we get from (4.28) that

$$f \circ F^{-1}(t)|C_n(t)| \le t^A(1-t)^A \|\gamma_n^U\|^2 o_{a.s.}(1). \tag{4.29}$$

Representation (4.21) and bounds (4.22), (4.24), (4.26), (4.29) taken together imply the bound

$$|\Upsilon_n(t)| \le ct^A(1-t)^A(\{\|R_n\| + \|R_n^U\|\}\|\beta_n^U\| + \{\|R_n\| + \|\beta_n^U\|\}\delta_n^{1/2+\epsilon_1} + \|\beta_n^U\|^2 o_{a.s.}(1)). \tag{4.30}$$

Lemma 4.5 is proved. □

Lemma 4.6 *Bound* (2.6) *holds true for all* $t \in (25\delta_n, \delta] \cup [1-25\delta_n, 1-\delta)$, *and any sufficiently small but fixed* $\delta > 0$.

Proof. It easy to prove (cf. M. Csörgő and Zitikis, 1996a, and Zitikis, 1998) that the bound $|V_n(t)| \le |\beta_n^U(t)||\gamma_n(t)|$ holds true for all $t \in (0,1)$, which in turn implies the bound

$$|\Upsilon_n(t)| \le |\beta_n^U(t)| f \circ F^{-1}(t)|\gamma_n(t)| + |\beta_n^U(t)|^2 \tag{4.31}$$

for all $t \in (0,1)$. Using bound (4.5) on the right-hand side of (4.31), we immediately obtain the following bound

$$\begin{aligned} |\Upsilon_n(t)| \le ct(1-t)\Big(&\sup_{t\in[25\delta_n, 1-25\delta_n]} \left|\frac{\gamma_n^U(t)}{\sqrt{t(1-t)}}\right| \sup_{t\in[25\delta_n, 1-25\delta_n]} \left|\frac{\beta_n^U(t)}{\sqrt{t(1-t)}}\right| \\ &+ \sup_{t\in[25\delta_n, 1-25\delta_n]} \left|\frac{\beta_n^U(t)}{\sqrt{t(1-t)}}\right|^2\Big)\{1 + o_{a.s.}(1)\}, \end{aligned} \tag{4.32}$$

which completes the proof of Lemma 4.6. □

Lemma 4.7 *Bound* (2.6) *holds true for all* $t \in [1/(n+1), 25\delta_n] \cup [1-25\delta_n, n/(n+1)]$.

Proof. Bounds (4.31) and (4.15) imply

$$\begin{aligned} |\Upsilon_n(t)| &\le ct^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}\delta_n^{1/2+\epsilon_1}|\beta_n^U(t)\{1 + o_{a.s.}(1)\} + c|\beta_n^U(t)|^2 \\ &\le t^{1-\epsilon}(1-t)^{1-\epsilon}\delta_n^{1/2+\epsilon_1} \sup_{t\in(0,1)} \left|\frac{\beta_n^U(t)}{t^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}}\right|\{1 + o_{a.s.}(1)\} \\ &\quad + t^{1-\epsilon}(1-t)^{1-\epsilon}\delta_n^{\epsilon_1} \sup_{t\in(0,1)} \left|\frac{\beta_n^U(t)}{t^{1/2-\epsilon/2}(1-t)^{1/2-\epsilon/2}}\right|^2. \end{aligned} \tag{4.33}$$

The proof of Lemma 4.7 is complete. □

Lemma 4.8 *Bound* (2.6) *holds true for all* $t \in (0, 1/(n+1)] \cup [n/(n+1), 1)$.

Proof. Let $t \in (0, 1/(n+1)]$. An application of bound (4.18) as well as $|\beta_n^U(t)| \le ct^{1-\lambda}\{1 + o_{a.s.}(1)\}$ (cf. the first inequality of (4.17)) on the right-hand side of (4.31) implies that the bound $|\Upsilon_n(t)| \le ct^{1-\lambda}\{t + t^\tau U_{1:n}^{1-\tau}\}\{1 + o_{a.s.}(1)\}$ holds true for any fixed $\lambda > 0$. Consequently, for any fixed $\epsilon > 0$ and $\lambda > 0$, we have the bound

$$|\Upsilon_n(t)| \le ct^{1-\epsilon}\{t^{1-\lambda+\epsilon} + t^{\tau-\lambda+\epsilon}U_{1:n}^{1-\tau}\}\{1 + o_{a.s.}(1)\}. \tag{4.34}$$

¿From now on throughout this proof we choose

$$\lambda = \epsilon/4.$$

Since the two statements

$$\limsup_{n\to\infty}\{n/\log n\}U_{1:n} < \infty$$

(cf. Exercise 2(i) on p. 408 of Shorack and Wellner (1986)) and

$$\liminf_{n\to\infty} n(\log n)^2 U_{1:n} = \infty$$

(cf. statement (3) on p. 408 of Shorack and Wellner (1986)) hold true almost surely, we derive from (4.34) that the bounds

$$\begin{aligned}|\Upsilon_n(t)| &\le ct^{1-\epsilon}\{(1/n)^{1-\lambda+\epsilon} + (1/n)^{\tau-\lambda+\epsilon}(1/n)^{1-\tau-\epsilon/4}\}o_{a.s.}(1)\\ &\le ct^{1-\epsilon}(1/n)^{1+\epsilon/2}o_{a.s.}(1)\\ &\le ct^{1-\epsilon}\delta_n^{1+\epsilon_1}o_{a.s.}(1)\end{aligned}$$

hold true for some $\epsilon_1 > 0$ and all $t \in (0, 1/(n+1)]$. In a similar way one proves that the bound $|\Upsilon_n(t)| \le c(1-t)^{1-\epsilon}\delta_n^{1+\epsilon_1}o_{a.s.}(1)$ holds true for all $t \in [n/(n+1), 1)$. Consequently, the bound

$$|\Upsilon_n(t)| \le ct^{1-\epsilon}(1-t)^{1-\epsilon}\delta_n^{1+\epsilon_1}o_{a.s.}(1) \tag{4.35}$$

holds true for all $t \in (0, 1/(n+1)] \cup [n/(n+1), 1)$. The proof of Lemma 4.8 is now complete. □

Proof of Theorem 2.3. Taking Lemmas 3.5-3.8 together, we complete the proof of Theorem 2.3. □

5 Concluding remarks

The current paper is a slight revision of our preprint Csörgő and Zitikis (1998). While preparing this paper for publication in this volume in Honour of Pál Révész, we were informed in February, 1999, about the existence of the paper Drees and de Haan (1999), where the authors have independently obtained results that are similar to ours, concerning the quantile process, under weaker assumptions than those spelled out in (1.13) and (1.14). Namely, instead of using the notion of regular variation, Drees and de Haan (1999) employ delicate modifications of this notion like, for example, $\mathcal{O}$-regular variation and extended regular variation. It would therefore be of interest to adapt these ideas of Drees and de Haan (1999) for the sake of modifying and improving other results of the present paper, and in particular those related to the Vervaat process V_n.

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