Fourier Analysis Applied to Super Stable and Related Processes

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Abstract

We use the Fourier transform to embed the $(\alpha, d, 1)$ superprocess in appropriate Sobolev spaces and obtain pathwise regularity results using maximal inequalities for the expected value of the supremum of "Ornstein-Uhlenbeck like" processes. Our techniques also give simple proofs of fluctuation theorems for the Brownian density process and a rescaling of the superprocess.

1 Hilbert Space Regularity Results

Among the results in a fundamental paper, Dawson [1972], the following situation was considered.

H is a Hilbert space with orthonormal basis $\{e_k\}_1^\infty$. A is a linear operator satisfying $Ae_k = -\lambda_k e_k$ where, for all large k, there exist $a, b, \delta > 0$ with $ak^{1+\delta} \leq \lambda_k \leq bk^{1+\delta}$. $\{b_k(t)\}_1^\infty$ is a sequence of standard Brownian motions and

$$X(t) = \sum_{k=1}^{\infty} x_k(t) e_k$$

where

$$x_k(t) = \int_0^t e^{-\lambda_k(t-s)} db_k(s)$$

is an Ornstein-Uhlenbeck process. Since $E[x_k^2(t)] = (1 - e^{-2\lambda_k t})/(2\lambda_k)$, $P(X(t) \in H) = 1$ for each fixed $t \geq 0$. In fact, noting $(I - A)^\gamma e_k = (1 + \lambda_k)^\gamma e_k$, it's true that $P((I - A)^\gamma X(t) \in H) = 1$ if $\gamma < \delta/[2(1+\delta)]$. Using a result of Newell [1962] which provides asymptotic estimates on the tail probabilities of 1-dimensional Ornstein-Uhlenbeck processes, Dawson showed that for any $\delta' < \delta$ and $T > 0$,

$$\sum_{k=1}^{\infty} P\left(\sup_{0 \leq t \leq T} x_k^2(t) \geq k^{-(1+\delta')}\right) < \infty;$$

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with Borel-Cantelli this shows that

$$P((I-A)^\gamma X \in C([0,\infty) : H)) = 1$$

if $\gamma < \delta/[2(1+\delta)]$.

In Blount and Bose [1997] the outline of Dawson's approach was applied to obtain Hilbert space regularity for the $(\alpha, d, 1)$ superprocess (see Dawson [1993]) by using Fourier transform techniques. The problem of having a continuous spectrum in contrast to the discrete case just considered was dealt with by breaking the parameter space into suitable blocks and obtaining a system of stochastic differential equations. These have a structure somewhat similar to the equations resulting from applying Ito's formula to $x_k^2(t)$, the square of an Ornstein-Uhlenbeck process. A maximal inequality was developed for the tail probabilities of the supremum over $t \in [0,T]$ of the processes; and a Borel-Cantelli argument yielded regularity results by embedding the Fourier transform of the superprocess in an appropriate Hilbert space. In this paper we reprove one of the results of Blount and Bose using a different argument (Theorem 1.1). We develop two maximal inequalities (Lemma 1.3) for the expected supremum of "Ornstein-Uhlenbeck like" processes and use these in place of estimating tail probabilities. We also prove a result showing on a pathwise basis that the superprocess cannot have point masses for $t > 0$. Although this has been proved by other means, it follows naturally from our results. In addition we rescale the superprocess to obtain a fluctuation theorem. We also prove a fluctuation theorem for the Brownian density process. Both of these are done very simply and naturally using Fourier transform techniques and Sobolev spaces as state spaces.

We now introduce some basic notation. Let $\mathcal{M}_\mathcal{F}$ be the space of finite positive measures on R^d topologized using the Prohorov metric, d (see Ethier and Kurtz [1986]). $C([0,\infty) : \mathcal{M}_\mathcal{F})$ is the set of continuous $\mathcal{M}_\mathcal{F}$-valued processes with topology defined by $\nu_n(\cdot) \to \nu(\cdot)$ if and only if $\sup_{0\le t\le T} d(\nu_n(t), \nu(t)) \to 0$ for each $T > 0$. Our probability space, (Ω, P), consists of $\Omega = C([0,\infty) : \mathcal{M}_\mathcal{F})$ with P being the distribution of the $(\alpha, d, 1)$ superprocess (which is subsequently defined).

For $x, \theta \in R^d$, let $e_{-\theta}(x) = e^{-i\theta\cdot x}$, and, for $\nu \in \mathcal{M}_\mathcal{F}$, $\hat{\nu}(\theta) = \nu(e_{-\theta}) = \int_{R^d} e_{-\theta}(x)\nu(dx)$. For $\gamma \in R$, let $H_\gamma = \{g : \int_{R^d} |g(\theta)|^2(1+|\theta|^2)^\gamma d\theta < \infty\}$; here $|\cdot|$ denotes the modulus of a complex number. $C([0,\infty) : H_\gamma)$ is the space of continuous H_γ-valued processes topologized by the seminorms $\sup_{0\le t\le T} \|h(t)\|_\gamma$ for $T > 0$. Note that $H_0 = L_2(R^d)$, and that for $\gamma < -(d/2)$, $C([0,\infty) : \mathcal{M}_\mathcal{F})$ embeds continuously into $C([0,\infty) : H_\gamma)$ by identifying $\{\nu(t)\}_{t\ge 0}$ with $\{\hat{\nu}(t,\theta)\}_{t\ge 0, \theta\in R^d}$. We use the notation $\nu(f) = \int_{R^d} f(x)\nu(dx)$ for $\nu \in \mathcal{M}_\mathcal{F}$ and f an appropriate function.

For $0 < \alpha \le 2$, let A_α be the fractional Laplacian satisfying $A_\alpha e_{-\theta} = -|\theta|^\alpha e_{-\theta}$ ($|\theta|$ is the Euclidean norm).

If $X(0) \in \mathcal{M}_\mathcal{F}$ and $E[X(0,1)] < \infty$, then for any bounded continuous function f with two bounded continuous derivatives, the $(\alpha, d, 1)$ superprocess has sample paths in $C([0,\infty) : \mathcal{M}_\mathcal{F})$ and solves the martingale problem

$$X(t,f) = X(0,f) + \int_0^t X(s, A_\alpha f)ds + M(t,f) \tag{1.1}$$

where $M(\cdot, f)$ is a continuous, square-integrable martingale with respect to $\sigma(X(s) : s \le t)$ and has quadratic variation process

$$[M(\cdot,f)](t) = \int_0^t X(s, f^2)ds. \tag{1.2}$$

Letting $\hat{X}(t,\theta) = X(t, e_{-\theta})$ and $\hat{M}(t,\theta) = M(t, e_{-\theta})$, we obtain

$$\hat{X}(t,\theta) = \hat{X}(0,\theta) - |\theta|^\alpha \int_0^t \hat{X}(s,\theta)ds + \hat{M}(t,\theta), \tag{1.3}$$

where $\hat{M}(t,\theta)$ is a complex martingale satisfying

$$[\mathrm{Re}\ \hat{M}(\cdot,\theta)](t) = \int_0^t X(s, \cos^2[\theta\cdot(\cdot)])ds \tag{1.4}$$

and

$$[\mathrm{Im}\ \hat{M}(\cdot,\theta)](t) = \int_0^t X(s, \sin^2[\theta\cdot(\cdot)])ds. \tag{1.5}$$

Using variation of constants we rewrite (1.3) as

$$\hat{X}(t,\theta) = e^{-|\theta|^\alpha t}\hat{X}(0,\theta) + \int_0^t e^{-|\theta|^\alpha (t-s)} d\hat{M}(s,\theta). \tag{1.6}$$

Letting $S_\alpha(t)$ be the Feller semigroup generated by A_α and noting $S_\alpha(t)e_{-\theta} = e^{-|\theta|^\alpha t}e_{-\theta}$, we also denote (1.6) by

$$X(t) = S_\alpha(t)X(0) + \int_0^t S_\alpha(t-s)dM(s), \tag{1.7}$$

and we set

$$Y(t) = \int_0^t S_\alpha(t-s)dM(s).$$

We can interpret each term in (1.7) as a measure (or signed measure) which has a Fourier transform given by the corresponding term in (1.6). Also (1.7) can be interpreted as holding in $C([0,T]:H_\gamma)$ for any $\gamma < -(d/2)$ and $T > 0$. Likewise (1.3) can be interpreted as

$$X(t) = X(0) + \int_0^t A_\alpha X(s)ds + M(t), \tag{1.8}$$

which holds in $C([0,T]:H_\gamma)$ for $\gamma < -(d/2)$ and $T > 0$; this is because the equation is an identity for the Fourier transforms, and all but the integral term are in this space (so all terms are). That M is in this space follows from Doob's quadratic maximal inequality and continuity in t (for fixed θ) of $\hat{M}(t,\theta)$. The regularity of (1.7) can be considerably improved using subsequent Lemma 1.3.

Note that $\hat{X}(t,\theta)$ is continuous in each variable separately. The following lemma shows $\hat{X}$ is jointly continuous in t and θ, and by (1.3) and (1.6) this also holds for $\hat{M}$ and $\hat{Y}$.

Lemma 1.1 *If $\nu \in C([t_1,t_2]:\mathcal{M}_\mathcal{F})$, then $\hat{\nu}(t,\theta)$ is jointly continuous in t and θ.*

Proof Consider

$$\begin{aligned} |\hat{\nu}(t,\theta) - \hat{\nu}(t_0,\theta_0)| &\leq |\hat{\nu}(t,\theta) - \hat{\nu}(t,\theta_0)| + |\hat{\nu}(t,\theta_0) - \hat{\nu}(t_0,\theta_0)| \\ &\leq \sup_{t_1\leq t\leq t_2} \nu(t, |e_{\theta-\theta_0} - 1|) + |\hat{\nu}(t,\theta_0) - \hat{\nu}(t_0,\theta_0)|. \end{aligned}$$

If $(t,\theta) \to (t_0,\theta_0)$, the second term goes to 0 by assumption. Note $\{\nu(t)\}_{t_1\leq t\leq t_2}$ is a compact set in $\mathcal{M}_\mathcal{F}$ and therefore is tight. Also,

$$|e_{\theta-\theta_0}(x) - 1| \leq (|\theta-\theta_0||x|) \wedge 2.$$

These facts imply the first term converges to 0 if $\theta \to \theta_0$. ∎

Lemma 1.2 *If $m(t)$ is a continuous, mean 0, martingale with quadratic variation process*

$$[m](t) = \int_0^t h(s)ds \le c(t),$$

where $c(t)$ is deterministic, then, for $a \ge 0$,

$$P(\sup_{0\le t\le \mu} |m(t)| \ge a) \le 2\exp\left(-\frac{a^2}{2c(\mu)}\right).$$

Proof For $\theta \in R$,

$$E\left[e^{\theta m(\mu)-(\theta^2 c(\mu)/2)}\right] \le E\left[e^{\theta m(\mu)-(\theta^2/2)\int_0^\mu h(s)ds}\right] = 1,$$

where the equality follows from Novikov's theorem. Thus $E[e^{\theta m(\mu)}] \le e^{\theta^2 c(\mu)/2}$.

If $\theta > 0$ and $a \ge 0$,

$$\begin{aligned} P\left(\sup_{0\le t\le\mu} m(t) \ge a\right) &\le E\left[e^{\theta m(\mu)}\right]e^{-\theta a} \\ &\le e^{((c(\mu)\theta^2)/2)-\theta a} \\ &\le \exp\left(\frac{-a^2}{2c(\mu)}\right), \end{aligned}$$

using Doob's submartingale inequality, Markov's inequality, and minimizing over θ. The same argument with $-m(t)$ proves the lemma. ∎

Lemma 1.3 *Assume $m(t)$ is a continuous, mean 0, martingale with quadratic variation process*

$$[m](t) = \int_0^t g(s)ds.$$

(a) If $\sup_{0\le s\le T} g(s) \le c$, a deterministic constant, then

$$y(t) = \int_0^t e^{-\beta(t-s)}dm(s)$$

satisfies

$$E\left[\sup_{0\le t\le T} y^2(t)\right] \le \begin{cases} 16cT & \text{if } \beta \ge 0 \\ \dfrac{8ce^{4T}\log(4\beta)}{\beta} & \text{if } \beta \ge 1. \end{cases}$$

(b) If $p \ge 1$ and $\sup_{0\le s\le T} E[g^p(s)] < \infty$, then

$$E\left[\sup_{0\le t\le T} y^{2p}(t)\right] \le \begin{cases} c(p)T^p \sup_{0\le t\le T} E[g^p(t)] & \text{if } \beta \ge 0 \\ \dfrac{c(p)e^{4pT}\sup_{0\le t\le T} E[g^p(t)]}{\beta^{p-1}} & \text{if } \beta \ge 1; \end{cases}$$

and

$$E\left[\sup_{0\le t\le T} y^2(t)\right] \le c(p,T)\sup_{0\le t\le T}(E[g^p(t)])^{1/p}(1\wedge\beta^{-(1-(1/p))}).$$

Proof Consider $y(t)$ restricted to $a \le t \le b$ and note

$$y(t) = e^{-\beta t}\Big[\int_0^a e^{\beta\mu} dm(\mu) + \int_a^t e^{\beta\mu} dm(\mu)\Big].$$

Thus

$$(*) \qquad \sup_{a\le t\le b} |y(t)| \le \left|\int_0^a e^{-\beta(a-\mu)} dm(\mu)\right| + \sup_{a\le t\le b}\left|\int_a^t e^{\beta(\mu-a)} dm(\mu)\right|.$$

First we prove (a).

Applying Lemma 1.2 to each term shows

$$\begin{aligned} P\left(\sup_{a\le t\le b}|y(t)|\ge q\right) &\le 2\left[\exp\left[\frac{-(q/2)^2}{2c\left(\frac{1-e^{-2\beta a}}{2\beta}\right)}\right] + \exp\left[\frac{-(q/2)^2}{2c\left(\frac{e^{2\beta(b-a)}-1}{2\beta}\right)}\right]\right] \\ &\le 4\exp\left[\frac{-\beta q^2}{4ce^{2\beta(b-a)}}\right]. \end{aligned}$$

Assume $\beta \ge 1$ and choose n such that $n \le \beta < n+1$. Set $t_k = (kT/n)$ for $0 \le k \le n$. Then our last result implies

$$P\left(\sup_{t_k\le t\le t_{k+1}}|y(t)|\ge q\right) \le 4\exp\left[\frac{-\beta q^2}{4ce^{4T}}\right].$$

Thus, since $n \le \beta$,

$$\begin{aligned} P\left(\sup_{0\le t\le T}|y(t)|\ge q\right) &\le \sum_{k=0}^{n-1} P\left(\sup_{t_n\le t\le t_{k+1}}|y(t)|\ge q\right) \\ &\le 4\beta\exp\left[\frac{-\beta q^2}{4ce^{4T}}\right]. \end{aligned}$$

We have, for any $q \ge 0$,

$$\begin{aligned} E\left[\sup_{0\le t\le T} y^2(t)\right] &= \int_0^\infty P\left(\sup_{0\le t\le T} y^2(t)\ge a\right) da \\ &\le \int_0^q 1 da + \int_q^\infty 4\beta\exp\left[\frac{-\beta a}{4ce^{4T}}\right] da \\ &= q + 16ce^{4T}\exp\left[\frac{-\beta q}{4ce^{4T}}\right]. \end{aligned}$$

Minimizing over q (basic calculus) gives the bound of

$$E\left[\sup_{0\le t\le T} y^2(t)\right] \le 4ce^{4T}\left(\frac{1+\log(4\beta)}{\beta}\right) \le \frac{8ce^{4T}\log(4\beta)}{\beta}.$$

Now assume $\beta \ge 0$. Then, using integration by parts,

$$y(t) = m(t) - \beta\int_0^t m(s)e^{-\beta(t-s)}ds.$$

Thus $|y(t)| \leq 2\sup_{0\leq\mu\leq t}|m(\mu)|$ and the bound follows from Doob's maximal inequality. This proves (a).

Now consider (b) and assume $\beta \geq 1$. Applying Burkholder's inequality to $(*)$ followed by Jensen's inequality shows

$$\begin{aligned} E\left[\sup_{a\leq t\leq b}|y(t)|^{2p}\right] &\leq c(p)E\left[\left(\int_0^a e^{-2\beta(a-\mu)}g(\mu)d\mu\right)^p + \left(\int_a^b e^{2\beta(\mu-a)}g(u)du\right)^p\right] \\ &\leq c(p)\left[\left(\frac{1-e^{-2\beta a}}{2\beta}\right)^p \sup_{0\leq\mu\leq a} E[g^p(u)] + \left(\frac{e^{2\beta(b-a)}-1}{2\beta}\right)^p \sup_{a\leq\mu\leq b} E[g^p(u)]\right] \\ &\leq c(p)\sup_{0\leq u\leq b} E[g^p(u)]e^{2p\beta(b-a)}\beta^{-p}. \end{aligned}$$

Choosing n and $\{t_k\}_{k=0}^n$ as in the proof of (a), we obtain

$$E\left[\sup_{t_k\leq t\leq t_{k+1}} y^{2p}(t)\right] \leq c(p)\sup_{0\leq t\leq T} E[g^p(t)]e^{4pT}\beta^{-p}.$$

Thus, recalling $n \leq \beta$,

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} y^{2p}(t)\right] &\leq \sum_{k=0}^{n-1} E\left[\sup_{t_k\leq t\leq t_{k+1}} y^{2p}(t)\right] \\ &\leq c(p)e^{4pT}\sup_{0\leq t\leq T} E[g^p(t)]\beta^{-(p-1)}. \end{aligned}$$

Now assume $\beta \geq 0$. As observed in the proof of (a), $|y(t)| \leq 2\sup_{0\leq\mu\leq t}|m(\mu)|$ and the bound in (b) for $\beta \geq 0$ follows from Burkholder's inequality.

The inequality for $E[\sup_{0\leq t\leq T} y^2(t)]$ follows from the first result and Jensen's inequality. ∎

Lemma 1.4 *Assume $E[X(0,1)] < \infty$.*
(a) If $\tau_n = \inf\{t : X(t,1) \geq n\}$ for $n = 1,2,\ldots$, and $Y_n(t) = \int_0^t S_\alpha(t-s)dM(s\wedge\tau_n)$, then, for $T > 0$ and $\gamma < (\alpha-d)/2$,

$$\int_{R^d} E\left[\sup_{0\leq t\leq T}|\hat{Y}_n(t,\theta)|^2\right](1+|\theta|^2)^\gamma d\theta < \infty.$$

(b)

$$P\left(\int_{R^d}\sup_{0\leq t\leq T}|\hat{Y}(t,\theta)|^2(1+|\theta|^2)^\gamma d\theta < \infty\right) = 1$$

if $\gamma < (\alpha-d)/2$.

Proof From (1.1), since $A_\alpha 1 \equiv 0$,

$$E[M^2(t,1)] = E\left[\int_0^t X(s,1)ds\right] = tE[X(0,1)]$$

and

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} X(t,1)\right] &\leq E[X(0,1)] + 2\left(E\left[\sup_{0\leq t\leq T} M^2(t)\right]\right)^{1/2} \\ &\leq E[X(0,1)] + 4\sqrt{TE[X(0,1)]}, \end{aligned}$$

using Doob's inequality.

Thus $\tau_n \uparrow \infty$ almost surely.

Consider $\hat{Y}_n(t,\theta) = \int_0^t e^{-|\theta|^{\alpha(t-s)}} d\hat{M}(s \wedge \tau_n, \theta)$ and observe

$$[\mathrm{Re}\hat{M}(\cdot \wedge \tau_n, \theta)](t) = \int_0^t I_{[0,\tau_n]}(s) X(s, \cos^2[\theta \cdot (\cdot)]) ds.$$

Note the integrand is dominated by n. An analogous result holds for $[\mathrm{Im}\hat{M}(\cdot \wedge \tau_n, \theta)](t)$. Thus, by Lemma 1.3,

$$E\left[\sup_{0 \le t \le T} |\hat{Y}_n(t,\theta)|^2\right] \le nC(T)\left(1 \wedge \frac{\log(4|\theta|^\alpha)}{|\theta|^\alpha}\right).$$

This shows the integral in (a) is bounded by

$$nc(T,d)\int_0^\infty \left(1 \wedge \frac{\log r}{r^\alpha}\right)(1+r^2)^\gamma r^{d-1} dr,$$

which is finite if $\gamma < (\alpha - d)/2$. This proves (a).

Let $\gamma < (\alpha - d)/2$ and set

$$\begin{aligned} A_n &= \int_{R^d} \sup_{0 \le t \le T} |\hat{Y}_n(t,\theta)|^2 (1+|\theta|^2)^\gamma d\theta; \\ A &= \int_{R^d} \sup_{0 \le t \le T} |\hat{Y}(t,\theta)|^2 (1+|\theta|^2)^\gamma d\theta. \end{aligned}$$

By (a), $P(A_n < \infty) = 1$ for all n, and $\tau_n \uparrow \infty$ a.s. implies

$$P(A_n \ne A \text{ infinitely often}) = 0.$$

Thus $P(A < \infty) = 1$. ∎

Theorem 1.1 *Assume* $E[X(0,1)] < \infty$ *and* $T > 0$. *Then*

(a)

$$P(Y \in C([0,T] : H_\gamma)) = 1 \text{ if } \gamma < \frac{\alpha - d}{2}.$$

(b) $P(X \in C([0,T] : H_\beta) \cap C((0,T] : H_\gamma)) = 1$ *if* $\gamma < (\alpha - d)/2$ *and* $\beta < -(d/2)$.

Proof $X(t) = S_\alpha(t)X(0) + Y(t)$. Since $|e^{-|\theta|^\alpha t} X(0,\theta)| \le e^{-|\theta|^\alpha t} X(0,1)$ it follows easily that (b) holds with $S_\alpha(t)X(0)$ in place of X. It suffices now to prove (a).

Consider, for $0 \le s$, $t \le T$,

$$\|Y(t) - Y(s)\|_\gamma^2 = \int_{R^d} |\hat{Y}(t,\theta) - \hat{Y}(s,\theta)|^2 (1+|\theta|^2)^\gamma d\theta.$$

Almost surely, $\hat{Y}(\mu,\theta)$ is continuous in μ for each fixed θ. By (b) of Lemma 1.4 we can apply the dominated convergence theorem to the last integral and obtain

$$P\left(\lim_{s \to t} \|Y(t) - Y(s)\|_\gamma = 0, 0 \le s, t \le T\right) = 1. \quad ∎$$

By conditioning on $X(0)$, the assumption $E[X(0,1)] < \infty$ in Theorem 1.1 can be weakened to require $X(0) \in \mathcal{M}_{\mathcal{F}}$. We could also have used Lemma 1.3(b) in place of Lemma 1.3(a). In particular, if $E[X(0,1)^p] < \infty$ for $p \geq 1$, then applying Burkholder's inequality shows $\sup_{0\leq t\leq T} E[X(t,1)^p] < \infty$, and by Lemma 1.3(b),

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} ||Y(t)||_\gamma^2\right] &\leq \int_{R^d} E\left[\sup_{0\leq t\leq T} |\hat{Y}(t,\theta)|^2(1+|\theta|^2)^\gamma\right] d\theta \\ &\leq c(T,p)\int_{R^d}(1\wedge|\theta|^{-\alpha(1-(1/p))})(1+|\theta|^2)^\gamma d\theta \\ &< \infty \text{ if } \gamma < \frac{\alpha(1-(1/p))-d}{2}. \end{aligned}$$

By conditioning on $X(0)$, we may assume $X(0,1)$ has moments of all orders, and Theorem 1.1 would follow as before. ■

Note that Theorem 1.1 shows that for $d = 1$ and $\alpha > 1$, almost surely for all $t > 0$ X has a density that is somewhat "smoother" than functions in $H_0 = L^2(R)$. In subsequent Theorem 1.2 we show that almost surely the $(\alpha, d, 1)$ superprocess cannot have any point masses for $t > 0$ and any value of α and d. While this result would follow from known results on the Hausdorff dimension on the support of X, it may be of interest that it follows from basic harmonic analysis and the fact that

$$\int_{R^d} \sup_{s\leq t\leq T} |\hat{X}(t,\theta)|^2(1+|\theta|^2)^{-d/2}d\theta < \infty$$

almost surely for any $0 < s < T$. Also, as just discussed, the assumption $E[X(0,1)] < \infty$ could be replaced by $X(0) \in \mathcal{M}_{\mathcal{F}}$.

Lemma 1.5 (a) *If* $\nu \in C([s,t] : \mathcal{M}_{\mathcal{F}})$, *then*

$$\sup_{s\leq\mu\leq t}\sum_{a\in R^d}\nu(\mu,\{a\})^2 = \sup_{s\leq\mu\leq t}\lim_{N\to\infty}(2N)^{-d}\int_{[-N,N]^d}|\hat{\nu}(\mu,\theta)|^2 d\theta.$$

(b) *If*

$$\int_{R^d}\sup_{s\leq\mu\leq t}|\hat{\nu}(\mu,\theta)|^2(1+|\theta|^2)^{-d/2}d\theta < \infty,$$

then $\sup_{s\leq\mu\leq t}\sum_{a\in R^d}\nu(\mu,\{a\})^2 = 0$.

Proof Let $\nu \in \mathcal{M}_{\mathcal{F}}$. Note $|\hat{\nu}(\theta)|^2 = \int_{R^d\times R^d} e_\theta(y-x)\nu(dx)\nu(dy)$. Then Fubini's theorem shows

$$(2N)^{-d}\int_{[-N,N]^d}|\hat{\nu}(\theta)|^2 d\theta = \int_{R^d\times R^d} I_N(x,y)\nu(dx)\nu(dy)$$

where, for $x = (x_1,\ldots,x_d)$ and $y = (y_1,\ldots,y_d)$,

$$I_N(x,y) = \prod_{k=1}^d \frac{\sin(N(x_k-y_k))}{N(x_k-y_k)}$$

converges boundedly and pointwise to

$$I(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else.} \end{cases}$$

This proves (a). For any $k > 0$, the right hand side of the equality in (a) can be bounded by

$$\lim_{N\to\infty} (2N)^{-d} \int_{|\theta|\le k} \sup_{s\le\mu\le t} |\hat{\nu}(\mu,\theta)|^2 d\theta + c(d) \int_{|\theta|>k} \sup_{s\le\mu\le t} |\hat{\nu}(\mu,\theta)|^2 (1+|\theta|^2)^{-d/2} d\theta.$$

The limit term equals 0 and, under the assumption in (b), by choosing k large, the second term can be made arbitrarily small. This proves (b). ∎

Theorem 1.2 *If $E[X(0,1)] < \infty$ and we replace (Ω, P) by its completion, then*

$$P\left(\sum_{a\in R^d} X(t,\{a\})^2 = 0 \text{ for } t > 0\right) = 1.$$

Proof For $0 < s \le t \le T$,

$$\sup_{s\le t\le T} |\hat{X}(t,\theta)|^2 \le \frac{1}{2}\left(e^{-|\theta|^\alpha s} X(0,1) + \sup_{s\le t\le T} |\hat{Y}(t,\theta)|^2\right).$$

By Lemma 1.4(b),

$$(*) \quad P\left(\int_{R^d} \sup_{s\le t\le T} |\hat{X}(t,\theta)|^2 (1+|\theta|^2)^{-d/2} d\theta < \infty\right) = 1.$$

Thus, by Lemma 1.5,

$$(**) \quad P\left(\sup_{s\le t\le T} \sum_{a\in R^d} X(t,\{a\})^2 = 0\right) = 1,$$

since the event in $(**)$ contains the event in $(*)$ which is measurable by Lemma 1.1. The theorem follows by letting $s = (1/n)$, $T = n$ and letting $n \to \infty$. ∎

2 Weak Convergence Results

In this section we prove two fluctuation theorems using the techniques of Section 1. We rescale the $(\alpha, d, 1)$ superprocess to obtain the first result, and then examine the Brownian density process first studied by Ito [1983].

Let X be the $(\alpha, d, 1)$ superprocess, and for $\epsilon > 0$, let

$$X_\epsilon(t, f) = \epsilon^d X(\epsilon^{-\alpha} t, f(\epsilon\cdot)).$$

From (1.1), we obtain the analogue of (1.3),

$$\hat{X}_\epsilon(t,\theta) = \hat{X}_\epsilon(0,\theta) - |\theta|^\alpha \int_0^t \hat{X}_\epsilon(s,\theta) ds + \epsilon^{(d-\alpha)/2} \hat{M}_\epsilon(t,\theta), \tag{2.1}$$

where M_ϵ is a continuous, square-integrable martingale with respect to $\sigma(X_\epsilon(s) : s \le t)$ and

$$[M_\epsilon(\cdot, f)](t) = \int_0^t X_\epsilon(s, f^2) ds. \tag{2.2}$$

We make the following assumptions in order to prove a fluctuation theorem and law of large numbers.

$$d > \alpha, \quad 0 < \epsilon \leq 1 \ \text{ and } \ \epsilon \to 0. \tag{2.3}$$

Remarks If $d = \alpha$, then essentially the same process is again obtained. If $\epsilon \to 0$ and $d < \alpha$, then assuming conditions that force convergence of $X_\epsilon(0,1)$, only convergence to the 0 process (for $t > 0$) is obtained. This is because

$$X_\epsilon(t,1) = X_\epsilon(0,1) + \epsilon^{(d-\alpha)/2} M_\epsilon(t,1) \ \text{ and } \ [M_\epsilon(\cdot,1)](t) = \int_0^t X_\epsilon(s,1)ds.$$

Thus $X_\epsilon(t,1)$ is a critical branching diffusion, and, if $d < \alpha$, the variance becomes infinite as $\epsilon \to 0$, forcing extinction for $t > 0$. This can be seen from computing the Laplace transform of $X_\epsilon(t,1)$.

Lemma 2.1 *Assume* $p \geq 1$ *and* $\sup_\epsilon E[X_\epsilon^p(0,1)] \leq c$. *Then*

$$\sup_\epsilon E\left[\sup_{0\leq t\leq T} X_\epsilon^p(t,1)\right] \leq c(T,p).$$

Proof Setting $\theta = 0$ in (2.1) and using (2.2) and Burkholder's inequality, we obtain

$$E\left[\sup_{0\leq t\leq T} X_\epsilon^p(t,1)\right] \leq 2^{p-1}\left[E[X_\epsilon^p(0,1)] + c(p)E\left[\left(\epsilon^{d-\alpha}\int_0^t X_\epsilon(s,1)ds\right)^{p/2}\right]\right].$$

The result then follows from basic computations and Gronwall's inequality. ∎

By analogy with (1.7), we can write

$$X_\epsilon(t) = S_\alpha(t)X_\epsilon(0) + \epsilon^{(d-\alpha)/2}Y_\epsilon(t), \tag{2.4}$$

where

$$Y_\epsilon(t) = \int_0^t S_\alpha(t-s)dM_\epsilon(s).$$

Lemma 2.2 *If* $\sup_\epsilon E[X_\epsilon(0,1)] < \infty$, *then*

(a) *the distributions of* M_ϵ *are relatively compact on* $C([0,\infty) : H_\gamma)$ *if* $\gamma < -d/2$.

(b) *The distributions of* Y_ϵ *are relatively compact on* $C([0,\infty) : H_\gamma)$ *if* $\gamma < (\alpha - d)/2$.

Proof Let $F_t^\epsilon = \sigma(X_\epsilon(s) : 0 \leq s \leq t)$ and consider, for $0 \leq t \leq t+\mu \leq T$ and $\mu \leq \delta$,

$$\begin{aligned}
E[||M_\epsilon(t+\mu) - M_\epsilon(t)||_\gamma^2 | F_t^\epsilon] &= \int_{R^d} E[|\hat{M}_\epsilon(t+\mu,\theta) - \hat{M}_\epsilon(t,\theta)|^2|F_t^\epsilon](1+|\theta|^2)^\gamma d\theta \\
&= \int_{R^d}\int_t^{t+\mu} X_\epsilon(s,1)ds(1+|\theta|^2)^\gamma d\theta \\
&\leq E\left[\left(\sup_{0\leq s\leq T} X_\epsilon(s,1)\right)c(\gamma)\delta|F_t^\epsilon\right] \\
&= E[A_\epsilon(\delta,T)|F_t^\epsilon],
\end{aligned}$$

where $A_\epsilon(\delta,T) = \sup_{0\leq s\leq T} X_\epsilon(s,1)c(\gamma)\delta$ satisfies, by Lemma 2.1,

$$\lim_{\delta\to 0}\sup_\epsilon E[A_\epsilon(\delta,T)] = 0.$$

(a) then follows from Kurtz's tightness condition, Theorem 3.8.6 of Ethier and Kurtz [1986].

Let $\tau_n(\epsilon) = \inf\{t : X_\epsilon(t,1) \geq n\}$. Using Lemma 2.1 and Markov's inequality,

$$P(\tau_n(\epsilon) \leq T) = P(\sup_{0\leq t\leq T} X_\epsilon(t,1) \geq n) \leq \frac{C(T)}{n}.$$

Thus it suffices to prove (b) with

$$Y_{n,\epsilon}(t) = \int_0^t S_\alpha(t-s)dM_\epsilon(\tau_n(\epsilon)\wedge s)$$

in place of Y_ϵ. Consider, for $0 \leq t \leq t+\mu \leq T$ and $\mu \leq \delta$,

$$Y_{n,\epsilon}(t+\mu) - Y_{n,\epsilon}(t) = (S_\alpha(\mu) - I)Y_{n,\epsilon}(t) + \int_t^{t+\mu} S_\alpha(t+\mu-s)dM_\epsilon(\tau_n(\epsilon)\wedge s).$$

Thus,

$$\begin{aligned}
&E[||Y_{n,\epsilon}(t+\mu) - Y_{n,\epsilon}(t)||_\gamma^2|F_t^\epsilon] \\
&\leq \frac{1}{2}E\left[\int_{R^d}(e^{-|\theta|^\alpha\mu}-1)^2|\hat{Y}_{n,\epsilon}(t,\theta)|^2(1+|\theta|^2)^\gamma d\theta|F_t^\epsilon\right] \\
&\quad +\frac{1}{2}E\left[\int_{R^d}\left(\int_t^{t+\mu} e^{-2|\theta|^\alpha(t+\mu-s)}I_{[0,\tau_n(\epsilon)]}(s)X_\epsilon(s,1)ds\right)(1+|\theta|^2)^\gamma d\theta|F_t^\epsilon\right] \\
&\leq E[A_\epsilon(\delta,T)|F_t^\epsilon]
\end{aligned}$$

where

$$\begin{aligned}
A_\epsilon(\delta,T) &= \int_{R^d}(e^{-|\theta|^\alpha\delta}-1)^2\left(\sup_{0\leq t\leq T}|\hat{Y}_{n,\epsilon}(t,\theta)|^2\right)(1+|\theta|^2)^\gamma d\theta \\
&\quad +n\int_{R^d}\left(\frac{1-e^{-2|\theta|^\alpha\delta}}{2|\theta|^\alpha}\right)(1+|\theta|^2)^\gamma d\theta.
\end{aligned}$$

Using Lemma 1.3 exactly as in the proof of Lemma 1.4(a) shows

$$E\left[\sup_{0\leq t\leq T}|\hat{Y}_{n,\epsilon}(t,\theta)|^2\right] \leq na(T,\theta),$$

where

$$\int_{R^d} a(T,\theta)(1+|\theta|^2)^\gamma d\theta \leq C(T,\gamma).$$

Thus, $\lim_{\delta\to 0}\sup_\epsilon E[A_\epsilon(T,\delta)] = 0$ by the dominated convergence theorem and (b) also follows from Kurtz's tightness criterion. ∎

We can now prove a law of large numbers for X_ϵ.

Theorem 2.1 *Assume* $\sup_\epsilon E[X_\epsilon(0,1)] < \infty$. *Let* $\psi(0) \in \mathcal{M}_\mathcal{F}$ *be deterministic and assume* $X_\epsilon(0) \xrightarrow{P} \psi(0)$ *as* $\mathcal{M}_\mathcal{F}$ *valued random variables. Then for any* $T > 0$,

$$\sup_{0\leq t\leq T} ||X_\epsilon(t) - S_\alpha(t)\psi(0)||_\gamma \xrightarrow{P} 0$$

if $\gamma < -(d/2)$, *and, for any* $0 < s \leq T$,

$$\sup_{s\leq t\leq T} ||X_\epsilon(t) - S_\alpha(t)\psi(0)||_\gamma \xrightarrow{P} 0$$

if $\gamma < (\alpha - d)/2$.

Proof $||X_\epsilon(0)-\psi(0)||_\gamma \xrightarrow{P} 0$ if $\gamma < -(d/2)$. The result then follows from Lemma 2.2(b) and (2.4). ∎

We now prove a fluctuation theorem for X_ϵ. For a deterministic $\psi(0) \in \mathcal{M}_{\mathcal{F}}$, let

$$\psi(t) = S_\alpha(t)\psi(0), \tag{2.5}$$

and

$$V_\epsilon(t) = \epsilon^{(\alpha-d)/2}(X_\epsilon(t) - \psi(t)). \tag{2.6}$$

Then

$$V_\epsilon(t) = V_\epsilon(0) + \int_0^t A_\alpha V_\epsilon(s)ds + M_\epsilon(t), \tag{2.7}$$

and, from (2.4),

$$V_\epsilon(t) = S_\alpha(t)V_\epsilon(0) + Y_\epsilon(t). \tag{2.8}$$

For the moment (2.7) and (2.8) can be considered as identities for the Fourier transforms without regard to regularity in particular spaces. We can now state a fluctuation theorem for X_ϵ.

Theorem 2.2 *Assume* $\sup_\epsilon E[X_\epsilon(0,1)^{1+\delta}] < \infty$ *for some* $\delta > 0$, *and* $V_\epsilon(0) \xrightarrow{d} V'(0)$ *in* H_{γ_0} *for some* $\gamma_0 < -(d/2)$. *Then* $(V_\epsilon, M_\epsilon) \xrightarrow{d} (V, M)$ *in* $C([0,\infty) : H_{\gamma_0} \times H_\beta) \cap C((0,\infty) : H_\gamma \times H_\beta)$ *if* $\beta < -(d/2)$ *and* $\gamma < (\alpha - d)/2$ *and* (V, M) *has the following properties.*

M is a Gaussian martingale and martingale measure with respect to the filtration $\sigma(V(s) : s \le t)$. M has independent increments and, almost surely, sample paths in $C([0,\infty) : H_\beta)$ if $\beta < -(d/2)$. If f is a continuous and bounded function, then $M(t, f)$ has quadratic variation process

$$[M(\cdot, f)](t) = \int_0^t \psi(s, f^2)ds. \tag{2.9}$$

The equation

$$V(t) = S_\alpha(t)V(0) + \int_0^t S_\alpha(t-s)dM(s) \tag{2.10}$$

holds almost surely in $C([0,\infty) : H_{\gamma_0}) \cap C((0,\infty) : H_\gamma)$ *if* $\gamma < (\alpha - d)/2$.

The equation

$$V(t) = V(0) + \int_0^t A_\alpha V(s)ds + M(t) \tag{2.11}$$

holds almost surely in $C([0,\infty) : H_{\gamma_0})$. $V(0) \stackrel{d}{=} V'(0)$ *and* $V(0)$ *is independent of* M.

Proof By Lemma 2.2, (2.7), (2.8) and the continuous mapping theorem, (2.10) and (2.11) hold for any distributional limit of (V_ϵ, M_ϵ).

Suppose f is a C^∞ function with compact support, and consider $(V_\epsilon, M_\epsilon(\cdot, f))$ and

$$(V_\epsilon, M_\epsilon^2(\cdot, f) - \int_0^\cdot X_\epsilon(s, f^2)ds) = (V_\epsilon, M_\epsilon^2(\cdot, f) - \int_0^\cdot [\psi(s, f^2) + \epsilon^{(d-\alpha)/2}V_\epsilon(s, f^2)]ds).$$

By Lemma 2.1 and Burkholder's inequality, the martingales are uniformly integrable. By problem 7 in Chapter 7 of Ethier and Kurtz [1986], any distributional limit (V, M) will satisfy (2.9) with f; by standard arguments this extends to any continuous bounded function. Since M has deterministic quadratic variation, it's Gaussian with independent increments (Lévy's theorem); and the other properties follow from the quoted problem of Ethier and Kurtz. M is thus unique in distribution, but integration by parts shows

$$V(t) = S_\alpha(t)V(0) + M(t) + \int_0^t A_\alpha S_\alpha(t-s)M(s)ds.$$

Thus V and M are unique in distribution. ∎

We now prove a law of large numbers and fluctuation limit for the Brownian density process.

Let $\{B_k(t)\}_{k=1}^n$ be independent standard Brownian motions in R^d with some initial distribution, and let

$$X_n(t) = \frac{1}{n}\sum_{k=1}^n \delta_{B_k(t)} \tag{2.12}$$

where δ_a is the probability measure with unit mass at a. Let Δ be the Laplacian and f a continuous, bounded function with two bounded and continuous derivatives. By Ito's formula, X_n satisfies

$$X_n(t,f) = X_n(0,f) + \int_0^t X_n\Big(s, \frac{\Delta}{2}f\Big)ds + n^{-1/2}M_n(t,f) \tag{2.13}$$

where $M_n(\cdot, f)$ is a continuous martingale with respect to $\sigma(X_n(s) : s \le t)$ and has quadratic variation

$$[M_n(\cdot, f)](t) = \int_0^t X_n(s, |\nabla f|^2)ds; \tag{2.14}$$

here ∇f is the gradient of f.

Letting $f = e_{-\theta}$ we obtain

$$\hat{X}_n(t,\theta) = \hat{X}_n(0,\theta) - \frac{|\theta|^2}{2}\int_0^t \hat{X}_n(s,\theta)ds + n^{-1/2}\hat{M}_n(t,\theta), \tag{2.15}$$

where $\hat{M}_n$ is a complex martingale satisfying

$$[\mathrm{Re}\ \hat{M}_n(\cdot,\theta)](t) = |\theta|^2\int_0^t X_n(s, \sin^2[\theta\cdot(\cdot)])ds \tag{2.16}$$

and

$$[\mathrm{Im}\ \hat{M}_n(\cdot,\theta)](t) = |\theta|^2\int_0^t X_n(s, \cos^2[\theta\cdot(\cdot)])ds. \tag{2.17}$$

We also have

$$\hat{X}_n(t,\theta) = e^{-(|\theta|^2/2)t}\hat{X}_n(0,\theta) + \int_0^t e^{-(|\theta|^2/2)(t-s)}d\hat{M}_n(s,\theta). \tag{2.18}$$

Let $S(t)$ be the Feller semigroup generated by $\Delta/2$. Then as before we denote (2.15) and (2.18) by

$$X_n(t) = X_n(0) + \int_0^t \frac{\Delta}{2}X_n(s)ds + n^{-1/2}M_n(t) \tag{2.19}$$

and

$$X_n(t) = S(t)X_n(0) + n^{-1/2}Y_n(t) \tag{2.20}$$

where

$$Y_n(t) = \int_0^t S(t-s)dM_n(s).$$

Since $X_n(t,1) \equiv 1$, we do not need any moment assumptions. The proofs are essentially identical to the previous results of this section but simpler without the need to compute moments. Thus we largely just describe the results.

Lemma 2.3 (a) *The distributions of* M_n *are relatively compact on* $C([0,\infty);H_\gamma)$ *if* $\gamma < -(d/2)-1$.

(b) *The distributions of* Y_n *are relatively compact on* $C([0,\infty):H_\gamma)$ *if* $\gamma < -d/2$.

Proof Exactly as for Lemma 2.2 without the need for stopping times. ∎

Theorem 2.3 *If* $X_n(0) \xrightarrow{P} \psi(0)$ *in* $\mathcal{M}_\mathcal{F}$, *then for any* $T>0$,

$$\sup_{0\le t\le T} ||X_n(t) - S(t)\psi(0)||_\gamma \xrightarrow{P} 0$$

if $\gamma < -(d/2)$.

Proof This follows from (2.20) and Lemma 2.3(b). ∎

For deterministic $\psi(0) \in \mathcal{M}_\mathcal{F}$, let

$$\psi(t) = S(t)\psi(0), \tag{2.21}$$

and

$$V_n(t) = \sqrt{n}(X_n(t) - \psi(t)). \tag{2.22}$$

Then

$$V_n(t) = V_n(0) + \int_0^t \frac{\Delta}{2} V_n(s)ds + M_n(t) \tag{2.23}$$

and, from (2.20)

$$V_n(t) = S(t)V_n(0) + Y_n(t). \tag{2.24}$$

We now state a fluctuation theorem for V_n.

Theorem 2.4 *Assume* $V_n(0) \xrightarrow{d} V'(0)$ *in* H_{γ_0} *for some* $\gamma_0 < -(d/2)$. *Then* $(V_n, M_n) \xrightarrow{d} (V, M)$ *in* $C([0,\infty): H_{\gamma_0}\times H_\beta) \cap C((0,\infty): H_\gamma \times H_\beta)$ *if* $\beta < -(d/2)-1$ *and* $\gamma < -(d/2)$, *and* (V,M) *has the following properties.*

M is a Gaussian martingale with respect to the filtration $\sigma(V(s): s\le t)$. *M has independent increments and, almost surely, sample paths in* $C([0,\infty): H_\beta)$ *if* $\beta < (-d/2)-1$. *If f is continuous and bounded with one bounded, continuous derivative, then* $M(\cdot, f)$ *has quadratic variation*

$$[M(\cdot,f)](t) = \int_0^t \psi(s, |\nabla f|^2)ds.$$

The equation

$$V(t) = S(t)V(0) + \int_0^t S(t-s)dM(s) \tag{2.25}$$

holds almost surely in $C([0,\infty): H_{\gamma_0}) \cap C((0,\infty): H_\gamma)$ *if* $\gamma < -(d/2)$. *The equation*

$$V(t) = V(0) + \int_0^t \frac{\Delta}{2} V(s)ds + M(t) \tag{2.26}$$

holds almost surely in $C([0,\infty): H_\gamma)$ *if* $\gamma < -(d/2)-1$ *and* $\gamma \le \gamma_0$. $V(0) \stackrel{d}{=} V'(0)$ *and* $V(0)$ *is independent of* M.

Proof The proof follows from Lemma 2.3, (2.23) and (2.24), exactly as the proof of Theorem 2.2 used Lemma 2.2, (2.7) and (2.8). ∎

The scaling used for Theorems 2.1 and 2.2 is used for proving a fluctuation theorem and law of large numbers for a more complex particle model in Dawson, Fleischmann, and Gorostiza (1989). Their initial measure is infinite and the state space is the standard space of tempered distributions. One advantage of using the Sobolev-spaces $\{H_\gamma\}$ is that it avoids technical difficulties arising from the fact that $A_\alpha g$ is not rapidly decreasing if g is but $\alpha < 2$. Of course if the initial measure (such as Lebesgue measure) does not have finite total mass, its distributional Fourier transform cannot be interpreted as a function of θ. In Blount and Bose (1997), this problem was avoided by first applying a suitable weighting function of the form $\varphi_p(x) = (1 + |x|^2)^{-p}$, for $p > d/2$, to give the measure finite mass. The weighted measure satisfies an equation analogous to (1.3); but additional perturbation terms arise from applying "Leibnitz's" formula to $A_\alpha(\varphi_p e_{-\theta})$. Because of technical complexity we have not used this approach here, but there doesn't appear to be any reason why it wouldn't work for the results in section 2 with suitable initial measures of infinite total mass.

The Brownian density process was studied in Ito [1983]. As a special case of a Mckean-Vaslov limit it is examined in Kallianpur and Xiong [1995], example 8.5.3. It is also examined (and named) in Walsh [1986]. By using the spaces $\{H_\gamma\}$ we are able to obtain very precise results on the regularity of the approximating and limiting processes, and the convergence results become technically very simple.

Finally, we note that versions of Lemma 1.2 and Lemma 1.5a are standard in the literature.

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