Random Walking Around Financial Mathematics

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Summary. The 1997 Nobel Prize in Economics was awarded to Robert C. Merton and Myron S. Scholes who, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. We start with quotations from the related press release that we then relate to mathematics in a historical context. In order to better appreciate what it took in mathematics to make these studies feasible at all, taking a historical scenic route, we review some of the fundamental notions of Brownian motion–Wiener process in Section 2, as well as some elements of Itô calculus in Section 3. Consequently, in Section 4, on assuming that stock prices fluctuate like geometric Brownian motion, we summarize the derivation of the Black–Scholes PDE and fair price formulae for European type options via the so–called risk–free portfolio method, though we are aware of the fact that the latter method is unquestionable only in a discrete–time setting. In Section 5 we discuss the problem of estimating the volatility parameter in the Black–Scholes formula, where the latter is also highlighted in terms of a geometric fractional Brownian motion. Section 6 deals with strong approximations of the logarithm of integral functionals of various geometric stochastic processes (Asian price type processes) by appropriate sup functionals of the processes in the exponents of their respective integrands. Whenever it seemed natural and desirable, we took the liberty of quoting some fundamental papers and books in full, throughout the text. All these are repeated in the References as well.

Keywords. Black–Scholes PDE and formula, risk–free portfolio, Feynman–Kac type forms, Brownian motion–Wiener process, Itô calculus, Itô formula, geometric Brownian motion, geometric fractional Brownian motion, volatility, integral and sup functionals, geometric stochastic processes in general, strong approximations, strong theorems, LIL and strong increments.

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1. Introduction

We start with quotations from the Press Release: **The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel,1997** `http://www.nobel.se/announcement-97/economy97.html`

> The Royal Swedish Academy of Sciences has decided to award the **Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, 1997**, to Professor **Robert C. Merton**, Harvard University, Cambridge, USA and Professor **Myron S. Scholes**, Stanford University, Stanford, USA *for a new method to determine the value of derivatives.* Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.
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> In a modern market economy it is essential that firms and households are able to select an appropriate level of risk in their transactions. This takes place on financial markets which redistribute risks towards those agents who are willing and able to assume them. Markets for options and other so-called derivatives are important in the sense that agents who anticipate future revenues or payments can ensure a profit above a certain level or insure themselves against a loss above a certain level. (Due to their design, options allow for hedging against one-sided risk – options give the right, but not the obligation to buy or sell a certain security in the future at a prespecified price.) A prerequisite for efficient management of risk, however, is that such instruments are correctly valued, or priced. A new method to determine the value of derivatives stands out among the foremost contributions to economic sciences over the last 25 years.
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> This year's laureates, **Robert Merton** and **Myron Scholes**, developed this method in close collaboration with **Fischer Black**, who died in his mid-fifties in 1995. These three scholars worked on the same problem: option valuation. In 1973, Black and Scholes published what has come to be known as the **Black-Scholes formula**. Thousands of traders and investors now use this formula every day to value stock options in markets throughout the world. Robert Merton devised another method to derive the formula that turned out to have very wide applicability; he also generalized the formula in many directions.
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> Black, Merton and Scholes thus laid the foundation for the rapid growth of markets for derivatives in the last ten years. Their method has more general applicability, however, and has created new areas of research - inside as well as outside of financial economics. A similar method may be used to value insurance contracts and guarantees, or the flexibility of physical investment projects.
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> **The Problem**
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> Attempts to value derivatives have a long history. As far back as 1900, the French mathematician Louis Bachelier reported one of the earliest attempts in his doctoral dissertation, although the formula he derived was flawed in several ways. Subsequent researchers handled the movement of stock prices and interest rates more successfully. But all of these attempts suffered from the same fundamental shortcoming: risk premia were not dealt with in a correct way.

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The method

Black, Merton and Scholes made a vital contribution by showing that it is in fact not necessary to use any risk premium when valuing an option. This does not mean that the risk premium disappears; instead it is already included in the stock price.

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One can use this argument, along with some technical assumptions, to write down a partial differential equation. The solution to this equation is precisely the Black-Scholes' formula. Valuation of other derivative securities proceeds along similar lines.

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The Black-Scholes formula

Black and Scholes' formula for a European call option can be written as

$$C = SN(d) - Le^{-rt}N\left(d - \sigma\sqrt{t}\right)$$

where the variable d is defined by

$$d = \frac{ln\frac{S}{L} + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

According to this formula, the value of the call option C, is given by the difference between the expected share value – the first term on the right-hand side – and the expected cost – the second term – if the option right is exercised at maturity. The formula says that the option value is higher the higher the share price today S, the higher the volatility of the share price (measured by its standard deviation) sigma, the higher the risk-free interest rate r, the longer the time to maturity t, the lower the strike price L, and the higher the probability that the option will be exercised (the probability is evaluated by the normal distribution function N).

Further Reading

Additional background material on the Bank of Sweden Prize in Economics Sciences in Memory of Alfred Nobel 1997, The Royal Swedish Academy of Sciences:
`http://www.kva.se/ecoback97.html`

Black, F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol. 81, pp.637–654.

Black, F., 1989, "How We came Up with the Option Formula", *The Journal of Portfolio Management*, Vol. 15, pp.4–8.

Hull, J.C., 1997, *Options, Futures and Other Derivates*, 3rd edition, Prentice Hall

Merton, R.C., 1973, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, Vol. 4, pp.637–654.

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We hope to be able to elucidate on the Black-Scholes formula in Sections 4 and 5. Without casting a shadow on the significance and importance of the matter of achievement in hand that has also inspired this exposition, we note in passing that, in this announcement of The Royal Swedish Academy of Sciences, **Louis Bachelier**'s "attempts" get quite a short shift indeed, to say the least. In order to put things to rights right at the outset, we quote here the Preface of Daniel W. Stroock to his Notes of April 1996, that deal with the same matter soundly, as well as in a correct historical context, *just about a year or so before* the announcement of this 1997 Nobel Prize in Economic Sciences. Stroock's introductory lines to his notes read as follows:

> "Itô's formula is now the bread and butter of the "quant" department of several major financial institutions. Actually, the application of what we now call *Brownian motion* to finance antedates its application to Brownian motion and goes back to the thesis, written at the turn of the century (five years before Einstein's famous paper about the kinetic theory of gases) by L. Bachelier. Bachelier was trying to model the fluctuations for prices on the Bourse. From a purely mathematical standpoint, his insights are far more penetrating than anyone else's prior to Wiener. In addition, their practical impact on all our lives is also far more penetrating. In fact, models, like that of Black and Scholes, which are the form in which Bachelier's ideas have been reincarnated, constitute the basis on which modern business makes decisions about how everything from stocks and bonds to pork belly futures should be priced. The role that Itô's formula plays in all this is very much the same as the one which it plays in our considerations. Namely, Itô's formula provides the link between various stochastic quantities and differential equations of which those quantities are the solution."

The present exposition is aimed at a wide audience in mathematics, and in the mathematical and economic sciences. Taking a historical route and, unfortunately, many more short cuts as well than one really should, we hope to succeed in at least illustrating what it took in more than the first half of the twentieth century, to create the mathematical foundations of the theory of probability and stochastic processes that has eventually also led to the possibility of modelling stock prices *correctly.* By writing *correctly* here, we only mean to say *mathematically correctly.* The question of a mathematically correctly posed model possibly being also correct from the point of view of the theory and practice of economic sciences is necessarily a different one of course. In this essay we will not deal with studying this problem.

We first consider a bank account or bond that is *risk free* in the sense that it yields r percents. More formally, let $\{\beta(t),\ 0 \le t \le T\}$ be the price of a *risk free* bond that yields at a *constant interest rate* r up to its time of maturity T. Then we have

$$\beta(t + \Delta t) - \beta(t) = r\beta(t)\Delta t$$

and, when compounded continuously,

$$d\beta(t) = r\beta(t)dt, \quad 0 \le t \le T. \tag{1.1}$$

At time $t \in [0, T]$

$$\beta(t) = \beta(0)e^{rt}, \quad \text{i.e.,} \quad \beta(0) = e^{-rT}\beta(T), \tag{1.2}$$

and, hence, we conclude

$$\beta(t) = e^{-r(T-t)}\beta(T), \quad 0 \le t \le T. \tag{1.3}$$

Now this *present value* $\beta(t)$ of a fixed rate risk free bond, that is the reduction of its value at maturity $\beta(T)$ at an appropriate scale, can be viewed as the *rational (fair) price* of $\beta(T)$ at time $t \in [0, T]$ when the time left to its maturity is $T - t$.

In the Black and Scholes (1973) and Merton (1973) model (cf. Remark 4.1) the *risky price* of a stock $\{S(t),\ 0 \le t \le T\}$ is "governed randomly" by a Brownian motion $\{W(t),\ 0 \le t < \infty\}$ via the Itô process:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad 0 \le t \le T, \tag{1.4}$$

i.e., by the stochastic process

$$S(t) = S(0) + \int_0^t rS(s)ds + \int_0^t \sigma S(s)dW(s), \quad 0 \le t \le T. \tag{1.5}$$

with positive constant coefficients r and σ.

On assuming some ideal market conditions, the aim of the game in a nutshell, in general, is to obtain a *rational (fair) price* formula for various options to buy or sell a certain security in the future at a prescribed price. As we will see (cf. (4.23), (4.25), (4.32), (4.33), (4.35) and (4.36)), under ideal market conditions, these formulae for various options are of the same general form that can, essentially, be viewed as "on an average imitations", under uncertainty, of the present value formula (1.3) of a fixed rate risk free bond.

Before we can however proceed any further, we first have to, and will attempt to, explain somewhat, what $W(\cdot)$ and (1.4), (1.5) are "all about". We hope to succeed in outlining some of the developments in mathematics that were initiated by

L. Bachelier (1900), Théorie de la spéculation, *Ann. Sci. École Norm. Sup.* **17**, 21-86,

the **first big step** in this regard. Indeed, Bachelier's models are based on random walks and their limiting cases, i.e., Brownian motions in contemporary language. The **next** 50 **or so years** produced some of the most impressive and strikingly NEW mathematics of the **twentieth** century that in the second half has acquired gigantic proportions and has gained immense importance as well.

Naturally, the literature on these developments is immense. The short "summaries" of our Sections 2 and 3, on Brownian motion and Itô calculus respectively, are not meant to be short introductions to, or reviews of, these huge areas of mathematics. The historical skeletal glimpse of these twentieth century branches of mathematics that we hope to give here is, at best, only an indicator of the vast background that precedes the "art" of financial modeling.

2. Brownian motion – Wiener process

Bachelier (1900), **Einstein** (1905) and **Smoluchowski** (1906) provided a theory of the peculiar erratic motion of small particles suspended in a liquid, first described in 1826 by the Scottish botanist **Brown**. In a series of papers beginning in 1920, **Wiener** undertook a mathematical analysis of Brownian motion. In his 1956 autobiography (pp. 38,39) Wiener writes:

> "The Brownian motion was nothing new as an object of study by physicists. There were fundamental papers by Einstein and Smoluchowski that covered it, but whereas these papers concerned what was happening to any given particle at a specific time, or the long-time statistics of many particles, they did not concern themselves with the mathematical properties of the curve followed by a single particle.
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> Here the literature was very scant, but it did include a telling comment by the French physicist Perrin in his book Les Atomes where he said in effect that the very irregular curves followed by particles in the Brownian motion led one to think of the supposed continuous non-differentiable curves of the mathematicians. He called the motion continuous because the particles never jump over a gap and non-differentiable because at no time do they seem to have a well-defined direction of movement."

Before continuing along these lines, we review briefly bits of the Bachelier (1900)–Einstein (1905)–Smoluchowski (1906) fundamental background. We begin with quoting from

Prize Presentation–Physics 1921

`hhttp://www.nobel.se/laureates/physics-1921-press.html`

Albert Einstein 1921 Nobel Laureate in Physics *for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect.*

Nobel Prize in Physics 1921. *Presentation Speech by Professor S. Arrhenius, Chairman of the Nobel Committee for Physics of the Royal Swedish Academy of Sciences.*

> Your Majesty, Your Royal Highnesses, Ladies and Gentlemen.
>
> There is probably no physicist living today whose name has become so widely known as that of Albert Einstein. Most discussion centres on his theory of relativity. This pertains essentially to epistemology and has therefore been the subject of lively debate in philosophical circles. It will be no secret that the famous philospher Bergson in Paris has challenged this theory, while other philosophers have acclaimed it wholeheartedly. The theory in question also has astrophysical implications which are being rigorously examined at the present time.
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> Throughout the first decade of this century the so-called Brownian movement stimulated the keenest interest. In 1905 Einstein founded a kinetic theory to account for this movement by means of which he derived the chief properties of suspensions, i.e. liquids with solid particles

suspended in them. This theory, based on classical mechanics, helps to explain the behaviour of what are known as colloidal solutions, a behaviour which has been studied by **Svedberg, Perrin, Zsigmondy** and countless other scientists within the context of what has grown into a large branch of science, colloid chemistry.
A third group of studies, for which in particular Einstein has received the Nobel Prize, falls within the domain of the quantum theory founded by Planck in 1900.

$\vdots$

An associated phenomenon is photo-luminescence, i.e. phosphorescence and fluorescence.

$\vdots$

Einstein's law of the photo-electrical effect has been extremely rigorously tested by the American Millikan and his pupils and passed the test brilliantly. Owing to these studies by Einstein the quantum theory has been perfected to a high degree and an extensive literature grew up in this field whereby the extraordinary value of this theory was proved. Einstein's law has become the basis of quantitative photo-chemistry in the same way as Faraday's law is the basis of electro-chemistry.*

* Being too remote from Sweden, Professor Einstein could not attend the ceremony.

The landmark 1905 paper of Einstein this Presentation Speech singles out is

A. Einstein (1905), On the movement of small particles suspended in a stationary liquid demanded by the molecular–kinetic theory of heat. *Ann. Physik* **17**, 549-560.

Let $\{W(t) = W(t, \omega);\ 0 \le t < \infty\}$ denote the Brownian motion of a particle ω as time t goes by. Then $W(t, \omega)$ represents the position of that particle at time t. The essential point in Einstein's 1905 modelling of $W(t)$ is that the contacts between the foreign microscopic particles and the particles of the liquid occur only at moments of collision. These collisions occur irregularly but often. Thus, if $t > s$ and the difference $t - s$ is large in comparison with the time interval between two successive collisions, then $W(t) - W(s)$ is the sum of a large number of small increments. Now, if the liquid is in macroscopic equilibrium, we may assume that the increments depend only on the length of their time interval and hence are homogeneous, and also that in disjoint time intervals they are independent. Assuming that Brownian motion has a probability density for the displacement of a particle within any fixed time interval and that this motion of a particle is symmetric, then the average increment over $t - s$ is $E(W(t) - W(s)) = 0$. Assume also that the average squared increment over $(t - s)$ is proportional to the length of this time interval, i.e., $E(W(t) - W(s))^2 = c(t - s) < \infty$, and that $E|W(t) - W(s)|^3 = o((t - s))$ as $(t - s) \to 0$.

Under these assumptions Einstein obtained the diffusion equation (2.2) and its fundamental solution (2.1) for the displacement of a Brownian particle. Namely, he derived the transition density for Brownian motion from the molecular theory of heat by concluding that

$$(2.1) \qquad p(t; x, y) := \frac{1}{dy} P_x(W(t) \in dy) \equiv \text{ the probability density function of a Brownian}$$

particle that starts from x and goes "into" y after a lapse of time $t > 0$

$$= \frac{1}{\sqrt{2\pi ct}} e^{-(y-x)^2/(2ct)}, \ t > 0, \ x, y \in \mathbb{R},$$

for each x, is the *fundamental solution* of the classical "heat" or "diffusion" partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} c \frac{\partial^2 p}{\partial y^2}, \tag{2.2}$$

i.e., right hand side expression of (2.1) is a solution of (2.2) for $t > 0$, and as $t \to 0$, the measures defined by using $p(t; x, \cdot)$ as a density (with respect to Lebesgue measure) converge weakly to a unit mass at x.

Moreover, Einstein also established a *relation* between c, some measurable (observable) parameters that are characteristic of a given system, and Avogadro's number N, the number of molecules in a **mole** ($N = 6.0248 \times 10^{23}$ **mole**$^{-1}$; **mole** := that quantity of substance whose mass (in grams) is numerically equal to the molecular weight of the substance), which in turn *led to an accurate method of measuring the latter by observing particles undergoing* Brownian *motion.*

Indeed, starting from x at $t = 0$, the mean-square displacement for being at $y \in \mathbb{R}$ at time t is given by the formula

$$E(W(t) - x)^2 = \int_{-\infty}^{\infty} (y - x)^2 \frac{1}{\sqrt{2\pi ct}} e^{-(y-x)^2/(2ct)} dy = ct. \tag{2.3}$$

Thus, according to Einstein's theory, if one observes a large number of free (no outside force, such as e.g. gravity) Brownian particles during the same time interval t and equates the empirically calculated mean-square deviation with the theoretically predicted value ct, with $c = 4kT/f$ (where T is the absolute temperature, f is the friction coefficient of the substance in hand (e.g., for spherical colloidal particles of radius r in a gas or liquid, f is given by Stokes' formula $f = 6\pi\eta rN$, where η is the viscosity coefficient), and k is the Boltzmann constant), then one can get an estimate of c and hence determine empirically the Avogadro number N. This, in turn, leads to *settling* the **Avogadro hypothesis** that *equal volumes of gases under the same conditions of pressure and temperature contain the same number of molecules.* This fundamental achievement may very well be the most successful application ever, of the so-called method of moments of the statisticians. We quote from M. Kac (1966, p. 54):

> "The successful determination of the Avogadro number from Brownian experiments was one of the great triumphs of Physics in the early days of the century and it dealt the final blow to the opponents of atomistic theories."

While Einstein could have of course received a Nobel Prize for several of his many singular achievements, we note in passing that, clearly, his fundamental 1905 work on Brownian motion had consequentially led to his 1921 Nobel Price. YET when **Statistical Quantum Mechanics** (Niels Bohr, 1922 Nobel Prize) and, as a consequence, the so–called **Uncertainty Principle** (Heisenberg, Schrödinger) entered the scene, Einstein countered by saying: "God does not roll dice".

With all due respect, one cannot but humbly add here that, though God was not rolling dice when "helping" to estimate the Avogadro number from Brownian experiments, He/She was certainly busy tossing a "fair" coin on that occasion. Indeed, one of the simplest models for Brownian motion that was already insightfully realized by Bachelier (1900) can be given in terms of the simple symmetric random walk model along the following lines.

Suppose that a particle is moving on the real line, starting from the origin. In each time unit it can only move one step to the right, or to the left, with probability one half and these steps are assumed to be independent. Say the i^{th} step of the particle is X_i. Then $X_1, X_2, \ldots$ are independent identically distributed random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}, \quad i = 1, 2, \ldots, \tag{2.4}$$

and after n steps the particle will be located at $S_n = X_1 + X_2 + \cdots X_n$. Refinements of the thus created path $S_1, S_2, \ldots$ imitate Brownian motion quite well if the time units and steps become progressively shorter (cf., e.g., P. Révész, 1990, Section 6.2).

Further to Bachelier's work, it is of interest to note also that he was first to establish the law of maximum displacement for Brownian motion, namely that we have

$$\sup_{0 \le s \le t} W(s) \stackrel{\mathcal{D}}{=} |W(t)| \quad \text{for each fixed } t > 0, \tag{2.5}$$

i.e.,

$$P_0\left\{\sup_{0 \le s \le t} W(s) \le y\right\} = 2\int_0^y \frac{e^{-u^2/(2ct)}}{\sqrt{2\pi ct}}du, \quad y \ge 0. \tag{2.6}$$

Physically speaking, in general, we can consider as Brownian motion the movement of any body which is subject to collision with other bodies, provided the dimension of the given body is small in comparison with the dimensions of the other bodies and if the contacts occur only at moments of collision and these collisions are of a random character.

In general, let $EW(t) = \mu t(-\infty < \mu < +\infty)$ and $E(W(t) - \mu t)^2 = \sigma^2 t$. As far as mathematical considerations go, we may assume without loss of generality that $\sigma^2 = 1$ and $\mu = 0$. Such a Brownian motion is called *normalized* (standard) *Brownian motion*, and this is the one this section is concerned with from now on. We note that $(W(t) - \mu t)/\sigma\sqrt{t}$ is a normalized Brownian motion if $W(t)$ is a Brownian motion with mean $EW(t) = \mu t$ and variance $E(W(t) - \mu t)^2 = \sigma^2 t$. We continue using the notation $W(\cdot)$ for a standard Brownian motion as well.

In a series of papers beginning in 1920 (cf. References) **Wiener** undertook a mathematical analysis of Brownian motion.

A stochastic process $\{W(t) = W(t, \omega); 0 \le t < \infty\}$, – where $\omega \in \Omega$, and $(\Omega, \mathcal{A}, P)$ is a probability space –, is called a standard Wiener process if

(a)
$$P(\omega : W(t,\omega) - W(s,\omega) \le x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{x} e^{-u^2/2(t-s)} du$$

for all $0 \le s < t < +\infty$ and $W(0) = 0$, i.e., the random variable $W(t,\omega) - W(s,\omega)$ is *normally distributed with mean* 0 *and variance* $t - s$, and we start $W(t)$ at $t = 0$ with probability one, and with $W(0) = 0$,

(b) $W(t)$ is an *independent increment process*, i.e., $W(t_2) - W(t_1),\ W(t_4) - W(t_3), \ldots, W(t_{2i}) - W(t_{2i-1})$ are independent random variables for all

$$0 \le t_1 < t_2 \le t_3 < t_4 \le \cdots \le t_{2i-1} < t_{2i} < \infty \ (i = 1, 2, \ldots),$$

i.e., by (a) and the notion of independence of events,

$$P\Big(\bigcap_{i=1}^{n} \{\omega : W(t_{2i},\omega) - W(t_{2i-1},\omega) \le x_i\}\Big)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_{2i} - t_{2i-1})}} \int_{-\infty}^{x_i} e^{-u^2/2(t_{2i}-t_{2i-1})} du, \quad \text{for any integer } \ n$$

(c) The *sample path function* $W(t,\omega)$ (i.e., $W(t,\omega)$ as a function of t for ω fixed) is everywhere *continuous in* t *with probability one* (i.e., except on an ω-set of P-measure zero $W(t,\omega)$ is an everywhere continuous function of t).

We note in passing that (a) and (b) imply $EW(s)W(t) = s \wedge t$ for all $s, t \in [0, \infty)$, and, conversely, $W(\cdot)$ is completely specified by its Gaussian finite dimensional distributions and its covariance structure.

Based on Bachelier (1900), Einstein (1905), Smoluchowski (1906, 1915, 1916), it was accepted that Brownian paths were governed by probabilistic laws as postulated in (a) and (b) above, and, ever since Bachelier, it was also believed that these paths were continuous. The problem of constructing a rigorous mathematical model that guarantees the existence of Brownian motion as in (a), (b) and (c), remained open.

Wiener (1920, 1923) provided the *first existence proof* via constructing *a mathematical model of Brownian motion* in which, based on the Daniell integral, *probabilities* were the *values* of *a measure on subsets* of a *space of continuous functions*, commonly called **Wiener measure** since.

Wiener's work in this regard constitutes a *first time measure theory* on a space of functions, more than a decade before **Kolmogorov**'s fundamental work,

Kolmogorov, A.N. *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933 and hence, naturally, without the help of Kolmogorov's extension theorem and that of **Doob**'s approach thirty or so years after,

Doob, J.L., *Stochastic Processes*, Wiley, New York, 1953.

For a short sketch of these measure theoretic considerations *à la* Kolmogorov (1933) combined with Doob (1953) *vis–à vis* the definition of a standard Wiener process, as well as for that of Wiener's Daniell integral based approach, we refer to M. Csörgő (1979, pp. 263-265). For more details in both directions, we refer to Doob (1966) and Kac (1966).

Wiener presented his fundamental results and ideas on Brownian motion in a series of papers published in the period from 1920 to 1923. In addition to the ones in 1920 and 1923 already mentioned here as well, Doob (1966) singles out Wiener (1921), where Wiener applied his measure to obtain a second model for Brownian motion that is essentially the model that was rediscovered by Uhlenbeck and Ornstein (1930). An account of his theories is included in Wiener (1930), as well as in a chapter of his 1934 book with **Paley**.

Moreover, in Paley and Wiener (1934) $W(t,\omega)$ is defined as a function on the unit interval with Lebesgue measure so that properties (a), (b), and (c) hold, i.e., a Wiener process on $C[0,1]$. This famous construction does not involve the Daniell integral, and $W(t,\omega)$ is represented explicitly as the sum of a Fourier series with random coefficients so that $W(\cdot,\omega)$ is a continuous function for almost all ω. Wiener's construction of Wiener measure via the Daniell integral is also restricted first to $C[0,1]$, but the simple transformation

$$W^*(t;\omega) := (1+t)\left(W\left(\frac{t}{t+1},\omega\right) - \frac{t}{t+1}W(1,\omega)\right), \quad 0 \le t < \infty,$$

yields $W^*(t;\omega)$ to be a Wiener process on $C[0,\infty)$ in both cases.

Having constructed his completely additive probability measure on $C[0,\infty)$, in his papers from 1920 to 1923, and also in his joint book with Paley in 1934, Wiener also studied the regularity of Brownian paths, finding *estimates* of the *moduli of continuity* for the path functions and proving that *almost all* $W(\cdot,\omega)$ *are nowhere differentiable*, and, even more strongly, that for every $\varepsilon > 0$ almost all $W(\cdot,\omega)$ satisfy the Lipschitz condition with exponent $1/2-\varepsilon$, but almost none with exponent $1/2+\varepsilon$.

P. Lévy (1937, 1939, 1940, 1948, 1950/51) had made fundamental lasting contributions to constructing and describing the fine analytic behaviour of Brownian path functions, such as their *modulus of continuity, local time, equivalence of local time, mesure du voisinage*, for example. Here we mention only P. Lévy's *modulus of continuity* that reads as follows: Almost all path functions $W(\cdot,\omega)$ are such that

$$(2.7) \qquad \lim_{h\downarrow 0} \frac{\sup_{0\le t\le 1-h}\sup_{0<s\le h}|W(t+s)-W(t)|}{(2h\log(1/h))^{1/2}} = \lim_{h\downarrow 0} \frac{\sup_{0\le t\le 1-h}|W(t+h)-W(t)|}{(2h\log(1/h))^{1/2}} = 1.$$

In M. Csörgő–P. Révész (1981) we *construct* a standard Wiener process $\{W(t,\omega);\ 0 \le t < \infty\}$ so that one has: For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that

$$P\left\{\sup_{0\le t\le 1-h}\sup_{0<s\le h}|W(t+s)-W(t)| \ge vh^{1/2}\right\} \le \frac{C}{h}e^{-v^2/(2+\varepsilon)} \tag{2.8}$$

for all $v > 0$ and $0 < h < 1$, and, using this inequality, we also prove (2.7)

As to how much non-differentiable are the paths of standard Brownian motion, we have (cf. M. Csörgő–P. Révész, 1979b, 1981) the following *modulus of non-differentiability*: Almost all path functions $W(\cdot,\omega)$ are such that

$$\lim_{h\downarrow 0}\inf_{0\le t\le 1-h}\sup_{0\le s\le h}\left(\frac{8\log(1/h)}{\pi^2 h}\right)^{1/2}|W(t+s)-W(t)| = 1. \tag{2.9}$$

This, in turn, implies of course that almost all path functions of $W(\cdot,\omega)$ are nowhere differentiable and characterizes also the Lipschitz nature of these paths as well.

The absence of differentiability of Brownian path is coupled with the *lack of times of increase* or *decrease* of the paths. The latter is perhaps one of the most intriguing aspects of Brownian sample path behaviour. We call $t > 0$ a time of increase (respectively decrease) of a function $f(t)$ if for some $\varepsilon > 0$, $f(t \mp h) \le f(t) \le f(t \pm h)$, $0 < h < \varepsilon$, respectively.

Dvoretzky, Erdős and Kakutani (1960/61) proved that, *with probability* 1, $W(t,\omega)$ *has no times of increase or decrease.*

The original proof of this fascinating theorem is demanding, to say the least. Based on P. Lévy's concept of Brownian local time, Knight (1981, p. 150) gives an elegant short proof of this subtle result. Moreover, he shows also that, if $(d/dt)W(t_0)$ is assumed to exist, then it is no loss of generality to assume also that $(d/dt)W(t_0) = c > 0$. But then t_0 is a time of increase, contradicting the just mentioned Dvoretzky-Erdős-Kakutani theorem. Thus we conclude again that $W(\cdot,\omega)$ is almost surely nowhere differentiable.

Proving non-differentiability this way (cf. also Geman and Horowitz, 1980) is a mathematical reincarnation of Perrin calling the motion of Brownian particles non-differentiable because at no time do they seem to have a well-defined direction of movement. So, indeed, Perrin was right about Brownian motion and mathematics gained a beauty, the continuous nowhere differentiable Wiener process, that has ever since been one of the most profound driving forces in our twentieth century mathematics that is also combined with an immense practical impact on all our lives.

One more strikingly beautiful property of Brownian motion before turning our attention to sketching some of the elements of stochastic calculus. We have the so-called Skorohod (1961) embedding scheme in mind, which essentially states that, knowing its past up to a given time, Brownian motion can stop and pretend to be the random variable associated with any distribution on the real line $I\!R$ that has at least two moments. A bit more precisely, for any distribution function F on $I\!R$ with first moment zero and finite second moment, one can define a probability space $(\Omega, \mathcal{A}, P)$ with a standard Wiener process $W(\cdot, \omega)$ and a finite stopping time $\tau \geq 0$ with finite expectation such that the distribution function of the random variable $W(\tau)$ is the given F. By saying that $\tau \geq 0$ is a stopping time, one means that the event $\{\omega : \tau(\omega) \leq t\}$ is an element of the σ–algebra generated by $\{W(s,\omega),\ s \leq t\}$. Thus, the simplest form of Skorohod's embedding theorem reads as follows.

Skorohod (1961): *Let $X : \Omega \to I\!R$ be an arbitrary random variable with distribution function F, $EX = \int_R x dF(x) = 0$, $EX^2 = \int_R x^2 dF(x) = 1$. There exists a probability space $(\Omega, \mathcal{A}, P)$ with a Wiener process $\{W(t);\ 0 \leq t < \infty\}$ and a finite stopping time random variable $\tau \geq 0$ such that*

$$X \overset{\mathcal{D}}{=} W(\tau) \quad \text{and} \quad E\tau = 1.$$

The proof of this theorem hinges on the strong Markovian property of $W(\cdot, \omega)$ (cf. K. Itô and H.P. McKean, Jr., 1974 and references therein): Suppose τ is an almost surely finite stopping time. Then $W(\tau+t) - W(\tau)$ is again a standard Brownian motion, and is independent of the pre–τ σ–algebra, just like as if τ were a non random fixed time.

3. Stochastic integration – Itô calculus

In the **classical calculus** of Newton and Leibniz the notion of the derivative, the slope of a curve at a point that in physics can be interpreted as the rate of change, takes precedence over that of the integral. Then, the fundamental theorem of calculus relates the integral as a differentiable curve "back" to the derivative.

Now to define rate of change in terms of Brownian motion directly, i.e., a stochastic "derivative" of some kind via that of Brownian motion, is meaningless *per se*, for Brownian motion cannot be differentiated. Hence, in **stochastic calculus**, the stochastic integral is defined first. Then the notion of stochastic differential is given meaning via using it (the integral) as a definition, *à la* the fundamental theorem of calculus, of the stochastic differential. Thus, a stochastic differential on its own has no meaning apart from that assigned to it when it enters a stochastic integral. For example, in the light of this "revelation", the meaning of the stochastic differential $dS(t)$ as "driven" by $dW(t)$ in (1.4) is determined via the stochastic integral for $S(t)$ in (1.5).

This assertion is however overcast for the moment by knowing that almost every sample path of $W(\cdot, \omega)$ has infinite variation on every finite interval, for if a sample function were to have bounded variation on any finite interval, then it would have a derivative existing almost everywhere there.

Nevertheless, *integration with respect to Wiener measure* has been an integral part of **Wiener**'s contributions right from the "beginning", i.e., Wiener (1923), where he introduced the stochastic integral $I(f,\omega) = \int f(t)d_tW(t,\omega) =: \int f(t)dW(t)$ in a somewhat indirect form (cf. Doob, 1953, p. 635). Here f is Lebesgue measurable and square integrable on $(-\infty,\infty)$, $f \in L^2(-\infty,\infty)$, and the standard Wiener process $\{W(t,\omega);\ 0 \le t < \infty\}$ is extended to the real line by letting $W(t,\omega) = \widetilde{W}(-t,\omega)$ for $t < 0$, where $\{\widetilde{W}(t,\omega); 0 \le t < \infty\}$ is another standard Wiener process that is independent of $W(t,\omega)$. For further first time contributions along these lines, we refer to Paley, Wiener and Zygmund (1933). For a discussion of Wiener type stochastic integrals as limits in L^2 of sequences of sums, we refer to Doob (1953, IX.2). Doob (1966, pp. 70–72) notes that the stochastic integral $I(f,\omega)$ was one of Wiener's most fruitful ideas that was for him (cf., e.g., Paley and Wiener, 1934) and still remains a fundamental tool in a variety of contexts. Moreover, analogous to $I(f,\omega)$, Wiener (1938) introduced multiple stochastic integrals and used them as a fundamental tool for studying *polynomial chaoses* as approximations to very general stationary processes.

In parallel to all these developments, the Einstein (1905) and Smoluchowski (1906, 1915) "heat equation line" of approach to studying Markovian processes has culminated in two pioneering papers along these lines that, in turn, has led to many others after:

Kolmogorov, A.N. (1931), Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, *Math. Ann.* **104**, 415–458,

Feller, W. (1936), Zur Theorie der stochastischen Prozesse (Existenz und Eindeutigkeitssätze), *Math. Ann.* **113**, 113–160.

As mentioned already, the original derivation of the heat equation in (2.2) from probabilistic assumptions is due to Einstein. Kolmogorov's derivation of his famous *backward and forward equations* combined with Feller's existence proof and investigation of the relation between these two equations have initiated an intensive study of Markov processes with continuous sample paths and transition densities that satisfy these equations. In their introduction, D.W. Stroock and S.R.S. Varadhan (1979) write: "The study of diffusion via the backward equation has been one of the more powerful and successful approaches to the subject and we have included a sketch of this procedure in Chapters 2 and 3."

Motivated by the *analytical approach* of **Kolmogorov** and **Feller**, and wanting to have a more probabilistically satisfactory approach to diffusions, **P. Lévy** suggested the *methodology of stochastic differential equations*. This approach was carried out by **K. Itô** (1942, 1944, 1946, 1951) via defining *stochastic integral equations* (cf., e.g., (1.5)), and thus also *stochastic integrals with random integrands* and Wiener *differentials*:

Kiyosi Itô (1942), Differential equations determining Markov processes (in Japanese), *Zenkoku Shijō Sūgaku Danwakai* **1077**, 1352–1400,

(1944), Stochastic Integral, *Proc. Imperial Acad. Tokyo* **20**, 519–524, (1946), On a stochastic integral equation, *Proc. Imperial Acad. Tokyo* **22**, 32–35,

(1951), On stochastic differential equations, *Mem. Amer. Math. Soc.* **4**, 1–51.

Doob (1953, Chapter IX.5) was first to study the *stochastic integral as a martingale.* On page 635 of this book he writes: "The stochastic integral in §5 is a generalization of one defined by Itô (1944), who treated the case in which the $y(t)$ process is the Brownian motion process. The use of martingale theory makes it possible to construct a closed system of these stochastic integrals, so that the integral with a variable upper limit defines a process of the same type as the process providing the original differential element."

For an excellent survey of, and further developments on the Itô integral, we refer to McKean (1969) and Doob (1984, Chapter 2.VIII), and for stochastic integrals with differentials more general than Brownian motion differentials to Dellacherie and Meyer (1980), Chung and Williams (1983), and Karatzas and Shreve (1988). For an up-to-date appreciation of the interplay between Brownian motion and martingales, we refer to Doob (1984), and Revuz and Yor (1999).

On the next few pages we hope to gain some insight into the intrinsic nature of Itô integrals of the form

$$I(t,\omega)=\int_0^t f(s,\omega)d_sW(s,\omega)=:\int_0^t f(s)dW(s), \tag{3.1}$$

without detailing any of the possibilities mentioned for their construction.

To start with, we have

$$\int_0^t W(s)dW(s)=\frac{W^2(t)}{2}-\frac{t}{2} \tag{3.2}$$

that contains the non-classical "extra" term $-t/2$.

Let $S_0=0$, $S_n=\sum_{k=1}^n X_k$, $n\geq 1$, where $X_k=\pm 1$, $k=1,2,\ldots$, is any numerical sequence of ± 1. Then, *purely algebraically*, we have

$$\begin{aligned}\sum_{k=1}^n S_{k-1}X_k &= \sum_{k=1}^n S_{k-1}(S_k-S_{k-1}) \\ &= \frac{1}{2}\sum_{k=1}^n\left(S_k^2-S_{k-1}^2\right)-\frac{1}{2}\sum_{k=1}^n\left(S_k-S_{k-1}\right)^2=\frac{S_n^2}{2}-\frac{n}{2},\end{aligned} \tag{3.3}$$

a "*complete agreement*" with (3.1), *a discrete* "**Itô formula**" *without* any *probabilities* attached to it.

Assuming now that the sequence of partial sums $S_0=0$, $\{S_k\}_{k=1}^\infty$ is a simple symmetric random walk, i.e.,

$$P\{X_i=1\}=P\{X_i=-1\}=1/2,\quad i=1,2,\ldots, \tag{3.4}$$

then (3.3) continues to hold true of course. Now, in this random walk context the "integrand" S_{k-1} in (3.3) is *non–anticipating*, i.e., is independent of the future values X_k, $X_{k+1}, \ldots$, the *random "signed measures of integration"*. Clearly then, in this probabilistic context, (3.3) is a *discrete analogue* of **Itô's formula** in (3.2).

This *elegant idea* of a *simple symmetric random walk* as in (3.3) *imitating* the *appearance* of the Itô *integral* in (3.2) is due to Pál Révész who, after presenting it in a seminar in the eighties at the Technical University in Budapest, posed the problem of constructing similar discrete analogues for more general versions of Itô's formula, as well as that of using them to approximate the latter via some embedding schemes. This program was successfully carried out by Szabados (1989/90, 1996).

Before sketching some of the further steps of this random walk approach to $I(t,\omega)$, we have a look again at (3.2) and (3.3). Assuming (3.4), we have

$$
\begin{aligned}
E\Big(\sum_{k=1}^{n} S_{k-1}X_k\Big)^2 &= \sum_{k=1}^{n}\Big(ES_{k-1}^2\Big)EX_k^2 \\
&= \sum_{k=1}^{n}(k-1) = \frac{n}{2}(n-1) = \frac{n^2}{2}\Big(1-\frac{1}{n}\Big),
\end{aligned}
\tag{3.5}
$$

which coincides of course with (cf. (3.3))

$$
E\Big(\frac{S_n^2}{2}-\frac{n}{2}\Big)^2 = \frac{n^2}{2}\Big(1-\frac{1}{n}\Big). \tag{3.6}
$$

In a similar vein, via Itô's isometry,

$$
\begin{aligned}
E\Big(\int_0^t W(s)dW(s)\Big)^2 &= E\Big(\int_0^t W^2(s)ds\Big) \\
&= \int_0^t EW^2(s)ds = \int_0^t sds = \frac{t^2}{2}
\end{aligned}
\tag{3.7}
$$

which, via (3.2), is of course the same as

$$
\begin{aligned}
E\Big(\frac{W^2(t)}{2}-\frac{t}{2}\Big)^2 &= \frac{t^2}{4}E\Big(\frac{W^2(t)}{t}-1\Big)^2 \\
&= \frac{t^2}{4}E\big(W^2(1)-1\big)^2 = \frac{t^2}{4}E\big(\chi_1^2-1\big)^2 = \frac{t^2}{2},
\end{aligned}
\tag{3.8}
$$

where χ_1^2 is chi-square random variable with one degree of freedom.

Thus, as expected, the "L^2 nature" of (3.3) is also similar to that of (3.2). Moreover, *purely algebraically again*, with $\varepsilon \in [0,1]$ we have

$$
\sum_{k=1}^{n}\big((1-\varepsilon)S_{k-1}+\varepsilon S_k\big)X_k \tag{3.9}
$$

$$
\begin{aligned}
&= (1-\varepsilon)\sum_{k=1}^{n} S_{k-1}\big(S_k - S_{k-1}\big) + \varepsilon \sum_{k=1}^{n} S_k\big(S_k - S_{k-1}\big)\\
&= (1-\varepsilon)\Bigg(\frac{1}{2}\sum_{k=1}^{n}\Big(S_k^2 - S_{k-1}^2\Big) - \frac{1}{2}\sum_{k=1}^{n}\big(S_k - S_{k-1}\big)^2\Bigg)\\
&\quad + \varepsilon\Bigg(\frac{1}{2}\sum_{k=1}^{n}\Big(S_k^2 - S_{k-1}^2\Big) + \frac{1}{2}\sum_{k=1}^{n}\big(S_k - S_{k-1}\big)^2\Bigg)\\
&= (1-\varepsilon)\Big(\frac{S_n^2}{2} - \frac{n}{2}\Big) + \varepsilon\Big(\frac{S_n^2}{2} + \frac{n}{2}\Big)\\
&= (1-\varepsilon)\ (\text{discrete forward “Itô formula” }) + \varepsilon\ (\text{discrete backward “Itô formula”})\\
&= \frac{S_n^2}{2} + \Big(\varepsilon - \frac{1}{2}\Big)n.\\
&= \begin{cases} \frac{S_n^2}{2} - \frac{n}{2}, & \text{if } \varepsilon = 0, \text{ a discrete (forward) “Itô formula”},\\ \frac{S_n^2}{2}, & \text{if } \varepsilon = \frac{1}{2}, \text{ a discrete “Fisk (1963,1966)–Stratonovich (1966) formula”},\\ \frac{S_n^2}{2} + \frac{n}{2}, & \text{if } \varepsilon = 1, \text{ a discrete backward “Itô formula” (cf. McKean (1969, p.35)).} \end{cases}
\end{aligned}
$$

Also, (cf. 2.29 Problem in Karatzas and Shreve (1988)) *working directly with* W and a *partition* Π of $[0,t]$ with $0 \le t_0 < t_1 < \ldots < t_m = t$, define the approximating sum

$$
I_\varepsilon(\Pi) := \sum_{i=0}^{m-1} \big[(1-\varepsilon)W(t_i) + \varepsilon W(t_{i-1})\big]\big(W(t_{i+1}) - W(t_i)\big)
$$

for the stochastic integral $\int_0^t W(s)dW(s)$. Then, with $||\Pi|| = \max_{1\le k\le m}|t_k - t_{k-1}|$, we have in L^2

$$
\lim_{||\Pi||\to 0} I_\varepsilon(\Pi) = \frac{1}{2}W^2(t) + \big(\varepsilon - \frac{1}{2}\big)t = \begin{cases} \text{the (forward) Itô integral if } \varepsilon = 0,\\ \text{the Fisk-Stratonovich integral if } \varepsilon = 1/2,\\ \text{the backward Itô integral if } \varepsilon = 1, \end{cases} \tag{3.10}
$$

and note that this limit is a martingale if and only if $\varepsilon = 0$, i.e., only when $W(\cdot)$ as an ‘integrand’ in $I_\varepsilon(\Pi)$ is evaluated at the left-hand end point of each interval $[t_i, t_{i+1}]$. The sensitivity of this limit to the value of ε is a consequence of the unbounded variation of the Brownian path.

Similarly to (3.2), if $f \in C^1(I\!R)$, then the (forward) Itô **formula** with the integrand $f(W(t))$ reads

$$
\int_0^t f(W(s))dW(s) = \int_0^{W(t)} f(s)ds - \frac{1}{2}\int_0^t f'(W(s))ds \tag{3.11}
$$

Let $g' = f$, i.e., $g := \big(\int f\big) \in C^2(I\!R)$. Then (3.11) reads

$$
g(W(t)) - g(W(0)) = \int_0^t g'(W(s))dW(s) + \frac{1}{2}\int_0^t g''(W(s))ds, \tag{3.12}
$$

formally leading to the **Itô chain rule** for differentials

$$
dg(W(t)) = g'(W(t))dW(t) + \frac{1}{2}g''(W(t))dt, \quad dt = (dW(t))^2, \tag{3.13}
$$

with the non-classical "extra" term $\frac{1}{2}g''(W(t))dt$. With $f(x)=x$, (3.11) implies (3.2), and (3.13) or (3.2) yields

$$d\Big(\frac{W^2(t)}{2}\Big) = W(t)dW(t) + \frac{1}{2}dt.$$

Purely algebraically again, we have the following discrete version of Itô's formula (Szabados, T., 1989/90, 1996):

$$\sum_{i=0}^{n-1} f(S_i)X_{i+1} = h(S_n) - \frac{1}{2}\sum_{i=0}^{n-1}\frac{f(S_{i+1})-f(S_i)}{X_{i+1}} \tag{3.14}$$

with any arbitrary function $f:\mathbb{Z}\to\mathbb{R}$, where

$$h(k)=\begin{cases}\frac{1}{2}f(0)+\sum_{j=1}^{k-1}f(j)+\frac{1}{2}f(k), & \text{if } k\ge 1,\\ 0, & \text{if } k=0,\\ -\frac{1}{2}f(0)-\sum_{j=1}^{-k-1}f(-j)-\frac{1}{2}f(k), & \text{if } k\le -1.\end{cases}$$

Proof (cf. also Lynn Kondo, 1994 for details). We have

$$h(S_{i+1})-h(S_i) = f(S_i)X_{i+1} + \frac{1}{2}\frac{f(S_{i+1})-f(S_i)}{X_{i+1}},$$

and summing both sides from $i=0$ to $i=n-1$ with $h(S_0)=g(0)=0$, we get (3.14).

Now if $S_0=0$, $\{S_k\}_{k=1}^{\infty}$ is a simple symmetric random walk, then (3.14) is a discrete version of Itô's formula in (3.11). Moreover, *à la* (3.9), with $\varepsilon\in[0,1]$, and f and h as in (3.14), we have also

$$\sum_{i=0}^{n-1}\big((1-\varepsilon)f(S_i)+\varepsilon f(S_{i+1})\big)X_{i+1} = h(S_n) + \big(\varepsilon-\frac{1}{2}\big)\sum_{i=0}^{n-1}\frac{f(S_{i+1})-f(S_i)}{X_{i+1}} \tag{3.15}$$

$$=\begin{cases} h(S_n)-\frac{1}{2}\sum_{i=0}^{n-1}\frac{f(S_{i+1})-f(S_i)}{X_{i+1}}, & \text{if } \varepsilon=0,\text{ a discrete (forward) Itô formula},\\ h(S_n), & \text{if } \varepsilon=\frac{1}{2},\text{ a discrete Fisk–Stratonovich formula},\\ h(S_n)+\frac{1}{2}\sum_{i=0}^{n-1}\frac{f(S_{i+1})-f(S_i)}{X_{i+1}}, & \text{if } \varepsilon=1,\text{ a discrete backward Itô formula}.\end{cases}$$

Now, as in the proof of Theorem 6 of Szabados (1996), we take a dyadic partition of the interval $[0,t]$, each subinterval of length 2^{-m}, where m is a nonnegative integer, and setting $s_m(0)=0$, we define the Skorohod stopping times

$$s_m(k)=\min\big\{s: s>s_m(k-1), |W(s)-W(s_m(k-1))|=2^{-m}\big\},\quad k\ge 1, \tag{3.16}$$

for an appropriately shrunk version of (3.15) (cf. Szabados, 1996, Lemma 11). Then, in this appropriately shrunk version of (3.15) the corresponding shrunk random walks can be replaced by $W(s_m(k))$, and, in lieu of the appropriately shrunk version of (3.15), with $\varepsilon \in [0,1]$ we get

$$(3.17)\quad \sum_{k=1}^{\lfloor t/\Delta s \rfloor} \left((1-\varepsilon) f\big(W(s_m(k-1)\big) + \varepsilon f\big(W(s_m(k))\big)\right)\big(W(s_m(k)) - W(s_m(k-1))\big)$$
$$= T_{x=0}^{W(s_m(\lfloor t/\Delta s \rfloor))} f(x)\Delta x + \Big(\varepsilon - \frac{1}{2}\Big) \sum_{k=1}^{\lfloor t/\Delta s \rfloor} \frac{f\big(W(s_m(k))\big) - f\big(W(s_m(k-1))\big)}{W(s_m(k)) - W(s_m(k-1))} \Delta s,$$

where $\Delta x = 2^{-m}$, $\Delta s = 2^{-2m}$, $\lfloor x \rfloor :=$ largest integer $\leq x$, and (cf. $h(\cdot)$ of (3.14))

$$(3.18)\quad T_{x=0}^{a} f(x)\Delta x = \varepsilon_a \Delta x \left\{ \frac{1}{2} f(0) + \sum_{j=1}^{(|a|/\Delta x)-1} f(\varepsilon_a j \Delta x) + \frac{1}{2} f(a) \right\},$$

with $a = W(s_m(\lfloor t/\Delta s \rfloor))$ and

$$(3.19)\quad \varepsilon_a = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0. \end{cases}$$

Assuming now that f is a continuously differentiable function on $\mathbb{R}$, $f \in C^1(\mathbb{R})$, then it follows from (3.17) by Theorem 6 of Szabados (1996) that, as $m \to \infty$, the indicated sums (random variables) converge with probability 1 for each $t > 0$, and we have

$$(3.20)\quad (1-\varepsilon)\int_0^t \overrightarrow{f(W(s))}\, dW(s) + \varepsilon \int_0^t \overleftarrow{f(W(s))}\, dW(s)$$
$$= \lim_{m\to\infty} \sum_{k=1}^{\lfloor t/\Delta s \rfloor} \left((1-\varepsilon) f\big(W(s_m(k-1)\big) + \varepsilon f\big(W(s_m(k))\big)\right)\big(W(s_m(k)) - W(s_m(k-1))\big)$$
$$= \lim_{m\to\infty} T_{x=0}^{W(s_m(\lfloor t/\Delta s \rfloor))} f(x)\Delta x + \Big(\varepsilon - \frac{1}{2}\Big) \lim_{m\to\infty} \sum_{k=1}^{\lfloor t/\Delta s \rfloor} \frac{f\big(W(s_m(k))\big) - f\big(W(s_m(k-1))\big)}{W(s_m(k)) - W(s_m(k-1))} \Delta s,$$
$$= \int_0^{W(t)} f(x)dx + \Big(\varepsilon - \frac{1}{2}\Big) \int_0^t f'(W(s))ds,$$

where $\int_0^t \overrightarrow{f(W(s))}\, dW(s)$ and $\int_0^t \overleftarrow{f(W(s))}\, dW(s)$ denote the (forward) Itô and backward Itô integrals, respectively.

With $\varepsilon = 1/2$, we have

$$(3.21)\quad \frac{1}{2}\left\{ \int_0^t \overrightarrow{f(W(s))}\, dW(s) + \int_0^t \overleftarrow{f(W(s))}\, dW(s) \right\} = \int_0^t f(W(s)) \circ dW(s) = \int_0^{W(t)} f(x)dx,$$

the Fisk–Stratonovich integral $\int_0^t f(W(s)) \circ dW(s)$ with the usual formula, while $\varepsilon = 0$, respectively $\varepsilon = 1$, give the (forward) Itô, respectively the backward Itô, formulae. Naturally, (3.20) also implies an almost sure (probability 1) version of (3.10).

With $g := (\int f) \in C^2(I\!R)$ as in (3.11), (3.20) reads

$$
\begin{aligned}
(3.22)\qquad & g(W(t)) - g(W(0)) \\
& = (1-\varepsilon)\int_0^t g'(\overrightarrow{W(s)})\, dW(s) + \varepsilon \int_0^t g'(\overleftarrow{W(s)})\, dW(s) - \Big(\varepsilon - \frac{1}{2}\Big)\int_0^t g''(W(s))ds,
\end{aligned}
$$

formally leading to the **combined forward–backward** Itô **chain rule**

$$
dg(W(t)) = (1-\varepsilon)g'(W(t))dW(t) + \varepsilon g'(W(t))dW(t) - \Big(\varepsilon - \frac{1}{2}\Big)g''(W(t))dt,
$$

with the non–classical "extra" term, $-(\varepsilon - \frac{1}{2})g''(W(t)dt$, unless $\varepsilon = 1/2$ that, in turn, yields the usual Fisk–Stratonovich chain rule (cf. (3.21)). With $\varepsilon = 0$, we get Itô's (forward) chain rule as in (3.13), and $\varepsilon = 1$ yields the backward Itô chain rule.

One of the many prominent examples of the use of Itô's integral in mathematics is a well–known unpublished result of Tanaka, the celebrated Tanaka formula (cf., e.g., McKean, 1969), that gives a representation of the local time process, $\eta(x,t)$, of a standard Wiener process, introduced by P. Lévy (1948), via an Itô integral. Namely, we have

Tanaka formula: *For any $x \in I\!R$ and $t > 0$, and for all $\omega \in \Omega$*

$$
(3.23)\qquad \eta(x,t) = |W(t) - x| - |x| - \int_0^t \operatorname{sign}(W(s) - x)dW(s),
$$

where $\eta(x,t)$ is defined by

$$
(3.24)\qquad H(A,t) := \int_A \eta(x,t)dx,
$$

i.e., the Radon–Nikodym derivative of the *occupation time* of W, $H(A,t)$, that in turn is defined by

$$
H(A,t) := \lambda\{s : s \le t,\ W(s) \in A\}
$$

for any Borel set A of the real line, where λ is the Lebesgue measure.

For the simple symmetric random walk $\{S_k\}_{k=1}^{\infty}$ as in (3.4), a natural definition of the *local time* process $\xi(x,n),\ n = 1,2,\ldots,$ is

$$
(3.25)\qquad \xi(x,n) := \#\{k : 0 \le k < n,\ S_k = x\}, \quad x = 0, \pm 1, \pm 2, \ldots.
$$

In order to better understand the intrinsic nature of Tanaka's formula (3.23) for Brownian local time, it is of interest to find a *discrete analogue* of this formula. We have (cf. Theorem 3 of M. Csörgő–P. Révész, 1985): *For any integer x, $n = 1,2,\ldots,$ and for all $\omega \in \Omega$,*

$$
(3.26)\qquad \xi(x,n) = |S_n - x| - |x| - \sum_{k=1}^{n-1} \operatorname{sign}(S_k - x)X_{k+1}.
$$

The **Tanaka formula** of (3.23) can be proved from an appropriately shrunk version of (3.26) via using strong approximation methods. We note also that the discrete Tanaka formula of (3.26) is a special case of the discrete Itô formula of (3.14) with $f(x) = \text{sign}(s - x)$.

4. On the Black–Scholes PDE and their risk–free portfolio method

Suppose at time $t = 0$ we sign a contract which gives us the *option* to buy, at a specific time T (called *maturity* or *expiration date*) one share of a stock at a specified price L (the so-called *exercise* or *strike price*).

If at maturity the price $S(T)$ of the stock is below the exercise price L then the contract is worthless to us. On the other hand, if $S(T) > L$, we can *exercise (call)* our *option* (realize the right to buy at the exercise price L) and then sell the share immediately in the market for $S(T)$.

This contract, which is an example of a *call option*, is thus equivalent to a payment of

$$(S(T) - L)^+ = \max(S(T) - L, 0)$$

dollars at maturity.

We list here some of the standard options (pay–off functions $f_t(\cdot),\ t \in [0, T]$), including the one that we have just used for illustration:

$(S(T) - L)^+ \equiv$ *European option* (option exercised at maturity $t = T$),

$\{(S(t) - L)^+,\ 0 \le t \le T\} \equiv$ *American option* (option exercised at any time between $t = 0$ and maturity $t = T$),

$\Big\{\Big(\max\limits_{t_0 \le s \le t} S(s) - L\Big)^+,\ 0 \le\ t_0 < t \le T\Big\} \equiv$ *Call on maximum (look back)* option,

$\Big\{\Big(\int_{t_0}^{t} S(s)ds - L\Big)^+,\ 0 \le t_0 < t \le T\Big\} \equiv$ *Call on average* (fixed strike price Asian) option.

Assume that the value V of a call option at any time $t \in [0, T]$ depends only on the underlying stock price $S(t)$ and the time t, i.e., we have

$$\{V = V(t, S(t)),\ 0 \le t \le T\}. \tag{4.1}$$

Moreover, assume that the stock price process $\{S = S(t),\ t \ge 0\}$ is governed (driven) by a standard Wiener process $\{W = W(t),\ 0 \le t < \infty\}$ on some probability space $(\Omega, \mathcal{A}, P)$ via the Itô process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad 0 \le t \le T, \tag{4.2}$$

where $\mu \in \mathbb{R}$ is a constant *appreciation rate* of the stock price, $\sigma > 0$ is a constant *volatility* coefficient, and $S(0) > 0$ is the initial stock price. Consequently, both processes are driven by the same Brownian motion.

Let $\mathcal{F}_t$, $0 \le t \le T$, be the σ–algebra generated by the values of a standard Wiener process $\{W(s),\ s \le t\}$ and completed by the addition of sets of P–probability zero. Thus $S(t)$ of (4.2) is $\mathcal{F}_t$–measurable, and hence so is also the value process V of (4.1) if it is smooth enough.

Assume further that the real valued function $V = V(t, S(t))$ on $[0,T] \times (0,\infty)$ is continuously differentiable in t and twice continuously differentiable in S, $V \in C^{1,2}[[0,T) \times (0,\infty)]$. Then V is also an Itô process (cf. Theorem B.1.1 in Musiela–Rutkowski, 1998), and by Itô's chain rule formula (cf. (3.13), or (3.22) with $\varepsilon = 0$, for intuition that amounts to saying that we have here a 'partial derivatives version' of (3.13) with respect to t (nonstochastic) and S (stochastic))

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2, \tag{4.3}$$

where dS is given by (4.2). Hence we have

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\mu S dt + \sigma S dW)^2 \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \\ &= \sigma S \frac{\partial V}{\partial S}dW + \Big(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt, \end{aligned} \tag{4.4}$$

where, for computing $(\mu S dt + \sigma S dW)^2$, we used the following "multiplication table" for differentials (cf. Karatzas–Shreve, 1988, p. 154)

	dt	dW
dt	0	0
dW	0	dt

It is easy to see that the geometric Brownian motion process

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}, \quad 0 \le t \le T, \tag{4.5}$$

is a solution of (4.2), starting from $S(0)$ at time 0. Indeed, on letting $F(t,W) := S(t)$, then Itô's formula (cf. (4.3)) for the process in (4.5) is

$$dS(t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial W}dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}(dW)^2. \tag{4.6}$$

By (4.5) we arrive at the differentials

$$\frac{\partial F}{\partial W} = \sigma S, \quad \frac{\partial^2 F}{\partial W^2} = \sigma \frac{\partial F}{\partial W} = \sigma^2 S, \quad \frac{\partial F}{\partial t} = \Big(\mu - \frac{1}{2}\sigma^2\Big)S,$$

which, in turn, via (4.6) yield (4.2) as desired, namely

$$\begin{aligned} dS(t) &= \sigma S(t) dW(t) + \Big(\mu - \frac{1}{2}\sigma^2\Big)S(t)dt + \frac{1}{2}\sigma^2 S(t)\underbrace{(dW(t))^2}_{dt} \\ &= \sigma S(t)dW(t) + \mu S(t)dt, \quad 0 \le t \le T. \end{aligned}$$

The uniqueness of the solution S in (4.5) for (4.2) follows from a general result of Itô, which states that a stochastic differential equation with Lipschitz continuous coefficients has a unique solution.

Now let us consider a *portfolio* $\Pi = \Pi(t)$ which involves *short selling* of one unit of a *European call option* and *long holding* θ units of the underlying asset. Using our general notation in (4.1), *the value of the portfolio* Π is given by

$$\Pi = -V + \theta S. \tag{4.7}$$

Assuming now that a bank account evolves at a riskless interest rate r as in (1.1), and that the stock price S is driven by a Brownian motion as in (4.2), we have (1.2) and (4.5) at our disposal.

The portfolio Π of (4.7) involves the unknown value V of a European call option driven by S via (4.2), and the unknown number θ times the price of a stock S at any time $t \in [0, T]$. Given (4.2), we have also (4.4). Consequently, in the light of having the dynamics of V as in (4.4) and the portfolio Π as in (4.7) with the *pay–off function*

$$f_T(S(T)) = (S(T) - L)^+, \tag{4.8}$$

the question is this: Given the form of the portfolio Π as in (4.7), can we determine V so that it should be a *fair (rational) price* for this *European call option*, say today at $t = 0$, or at any time $t \in [0, T]$?

The portfolio Π under consideration reflects a *hedge position* in that it combines an option with its underlying stock asset so that either the stock protects the option against loss, or vice versa. As Black (1989) puts it: "If the stock goes up, you will lose on the option but make it up on the stock. If the stock goes down, you will lose on the stock but make it up on the option." By adjusting the proportion of the option and stock *continuously* in a portfolio, Black and Scholes (1973) and Merton (1973) demonstrated that investors can create a *riskless hedging portfolio* in which all market risks are eliminated.

The model of a geometric Brownian motion as in (4.5) was suggested by Samuelson (1965). In combination with (1.1), it underlies the Black–Scholes model and the famous Black-Scholes formula for the fair (rational) price of a European call option with pay–off function f_T as in (4.8). We *emphasize* that we are *not concerned* here with the question of *whether* this *model is* the *correct* one for describing asset price fluctuations. Rather, we are only trying to gain some insight into the model via its mathematics.

The **Black** and **Scholes** (1973) formulation establishes the equilibrium condition between the expected return on the option, the expected return on the stock and the riskless interest rate under the following *hypotheses*:

(i) Stock price follows geometric Brownian motion with parameters μ and σ constants as in (4.2),

(ii) Trading takes place continuously in time,

(iii) The riskless interest rate r of a bank account or bond is constant over time, and investors can borrow or lend at the same risk–free rate of interest,

(iv) The asset (stock) pays no dividend during the lifetime of the option,

(v) There are no transaction costs in buying or selling the asset or the option, and there are no taxes,

(vi) Short selling is permitted without penalties,

(vii) There are no riskless arbitrage opportunities.

First we note that, given the assumptions (iv), (v) and (vi), the value formula of the portfolio Π in (4.7) is correct and, due to assumption (ii), the proportion of the option and stock in the portfolio can be readjusted continuously. Indeed, we have already assumed that the value V of a call option will be as in (4.1) and, moreover, so smooth that $V \in C^{1,2}\big[(0,T] \times (0,\infty)\big]$. Consequently, for V in (4.7) we already have (4.3), as well as (4.4), on assuming also (i).

As to *continuously adjusting* the proportion of the short European option and the number of θ units of the underlying asset in the portfolio Π, we will first keep θ *instantaneously a constant*, and *then*, for reasons to come, we will choose it to be equal to $\frac{\partial V}{\partial S}$, that is to say, we proceed via *first* taking the differential $d\Pi(t)$ to be

$$d\Pi = \theta dS - dV, \tag{4.9}$$

and *then* letting

$$\theta = \frac{\partial V}{\partial S}. \tag{4.10}$$

This riskless portfolio method, with (4.9) and (4.10) combined, will result in the correct form of the Black–Scholes Fundamental Equation (cf. (4.19)), and hence also leads to the correct form of the fair price for a European call option with pay–off function f_T as in (4.8). However, as stressed by M. Musiela and M. Rutkowski (1998), the risk–free portfolio approach, mathematically speaking, is unquestionable only in a discrete–time setting. We will return to this point later on (cf. Remark 4.2).

Presently we continue our discussion in the context of Itô calculus essentially along the initial main lines of the Black–Scholes (1973) paper that tacitly also assumes that their riskless hedging portfolio Π is *self–financing*, and *hence* satisfies the condition (cf. Section 5.2 of Musiela–Rutkowski, 1998)

$$d\Pi = \frac{\partial V}{\partial S} dS - dV. \tag{4.11}$$

First, with $d\Pi$ as in (4.9), via (4.2) and (4.4), we have

$$\begin{aligned} d\Pi &= \theta dS - dV \\ &= \theta(\mu S dt + \sigma S dW) - \sigma S \frac{\partial V}{\partial S} dW - \Big(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big) dt \\ &= \sigma S\Big(\theta - \frac{\partial V}{\partial S}\Big) dW + \mu S\Big(\theta - \frac{\partial V}{\partial S}\Big) dt - \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big) dt. \end{aligned} \tag{4.12}$$

Now, on choosing $\theta = \frac{\partial V}{\partial S}$ for continuously changing the proportion of the stocks in Π, its instantaneous rate of change becomes

$$d\Pi = -\Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt, \tag{4.13}$$

i.e., *non-stochastic* in t (the random term $\sigma S(\theta - \frac{\partial V}{\partial S})dW$ was removed, $d\Pi$ *is not a stochastic differential any more*) and, on account of having removed also the term $\mu S(\theta - \frac{\partial V}{\partial S})$, it does not any more contain the expected return μ on the stock price. Consequently, *risk preferences* of the investors are also *eliminated*, and the portfolio Π becomes *risk free* (a *riskless hedge*). Since we assume that there are no arbitrage opportunities allowed (cf. (vii) above), and that there are no dividends nor costs involved in 'continuously adjusting' the portfolio (cf. (iv) and (v) above), Π *should be just like money in the bank* (cf. (1.1)), i.e., it should grow like

$$d\Pi = r\Pi dt. \tag{4.14}$$

Indeed, otherwise, i.e., if a hedged portfolio were to earn more than the riskless interest rate r, then an arbitrageur could earn 'limitless' riskless profit by simply borrowing as much money as possible (cf. (iii)) to buy hedged portfolios.

Consequently, by (4.13) and (4.14) combined, we have

$$r\Pi dt = -\Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt, \tag{4.15}$$

with

$$\Pi = \frac{\partial V}{\partial S} S - V \tag{4.16}$$

by (4.7) and (4.10). Hence we obtain

$$r\Big(\frac{\partial V}{\partial S} S - V\Big)dt = -\Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt \tag{4.17}$$

and, on account of the dt differentials being common to all factors on both sides, we arrive at the partial differential equation (PDE)

$$r\Big(\frac{\partial V}{\partial S} S - V\Big) = -\Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big). \tag{4.18}$$

Conclusion 4.1. Rearranging the terms in (4.18), we obtain the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{4.19}$$

that is usually called the Black–Scholes PDE, or the Black–Scholes *Fundamental Equation*, which is *to be solved* now *subject to the terminal condition*

$$V(T, S(T)) = (S(T) - L)^+, \tag{4.20}$$

that is due to the pay–off function f_T as in (4.8), and also subject to the natural boundary condition

$$V(t, 0) = 0, \quad t \in [0, T]. \tag{4.21}$$

This equation has an explicit solution. It is given by

$$\begin{aligned} V(t, S(t)) &= S(t)\Phi(d) - Le^{-r(T-t)}\Phi(d - \sigma\sqrt{T-t}), \\ &=: C_t(S(t), L, T-t), \end{aligned} \tag{4.22}$$

where

$$d = \frac{ln\frac{S(t)}{L} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leq t \leq T,$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \quad x \in I\!R,$$

the unit normal distribution function.

Given the Black–Scholes model, this solution V of the Black–Scholes PDE that we have denoted by C_t is *the market value of*, a *fair (rational) price* for, a *European call option* at time $t \in [0, T]$ with *expiry date* T, *pay–off function* $f_T = (S - L)^+$ and *strike (exercise) price* L, when the time left till maturity is $T - t$.

On setting $t = 0$, we obtain the **Black–Scholes formula** as quoted in the Introduction of this exposition, with $C = C_0$, $S = S(0)$, $N := \Phi$. $\square$

Remark 4.1. Solving the Black–Scholes PDE (4.19) subject to (4.20) and (4.21) is *equivalent* to

$$\begin{aligned} V(t, S(t)) &= e^{-r(T-t)} E\big((S(T) - L)^+ \big| \mathcal{F}_t\big) \\ &= C_t(S(t), L, T-t), \end{aligned} \tag{4.23}$$

where C_t is as in (4.22), and $S(t)$ now is the solution of

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{4.24}$$

to begin with, i.e., instead of the equation (4.5) we start with

$$S(t) = S(0)e^{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)}, \tag{4.25}$$

where *we kept the same notation for the sake of simplicity.*

The latter is the so–called *discounted stock price* of the Black–Scholes model that eliminates the constant appreciation rate μ of the stock price (cf. (4.2)) by arriving at the Black–Scholes formula via (4.23) with $t = 0$. $\square$

The Feynman–Kac type version (4.23) of (4.22) is appealing. It is more informative of the nature of the solution of the Black–Scholes PDE (4.19), subject to (4.20), than its direct version (4.22) is. As an illustration of the use of (4.23) for calculating the Black–Scholes formula, we derive its form via calculating $C_0(S(0), L, T)$ directly from it. Namely, we *obtain*

$$\begin{aligned} C_0(S(0), L, T) &= e^{-rT} E\big((S(T) - L)^+ \big| \mathcal{F}_0\big) \\ &= e^{-rT} E\big((S(T) - L)^+\big) \\ &= S(0)\Phi(d) - Le^{-rT}\Phi(d - \sigma\sqrt{T}), \end{aligned} \tag{4.26}$$

where

$$d = \frac{ln\frac{S(0)}{L} + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}. \tag{4.27}$$

Proof. We start with (cf. (4.25) with $t = T$)

$$\begin{aligned} X = X(T) := ln\frac{S(T)}{S(0)} &= \big(r - \frac{1}{2}\sigma^2\big)T + \sigma W(T) \\ &\overset{\mathcal{D}}{=} N\Big(\big(r - \frac{1}{2}\sigma^2\big)T, \sigma^2 T\Big), \end{aligned} \tag{4.28}$$

and consider

$$\begin{aligned} E(S(T) - L)^+ &= \int_{\{S(T)>L\}} (S(T) - L)dP \\ &= \int_{\{S(T)>L\}} S(T)dP - LP\{S(T) > L\} \\ &=: I - LP\{S(T) > L\}, \end{aligned} \tag{4.29}$$

where P here stands for the normal distribution of the random variable X as in (4.28).

Calculating now the first term I of (4.29), we have

$$\begin{aligned} I &= S(0)\int_{\left\{ln\frac{S(T)}{S(0)} > ln\frac{L}{S(0)}\right\}} \exp\Big(ln\frac{S(T)}{S(0)}\Big)dP \\ &= S(0)\int_{ln\frac{L}{S(0)}} e^y \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left\{-\frac{\big(y - \big(r - \frac{1}{2}\sigma^2\big)T\big)^2}{2\sigma^2 T}\right\}dy \\ &= S(0)e^{rT}\frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{ln\frac{L}{S(0)}} \exp\left\{-\frac{\big(y - \big(r + \frac{1}{2}\sigma^2\big)T\big)^2}{2\sigma^2 T}\right\}dy \\ &= S(0)e^{rT}\frac{1}{\sqrt{2\pi}}\int_{\frac{ln\frac{L}{S(0)} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} e^{-y^2/2}dy \\ &= S(0)e^{rT}\frac{1}{\sqrt{2\pi}}\int_{-d} e^{-y^2/2}dy \\ &= S(0)e^{rT}(1 - \Phi(-d)) = S(0)e^{rT}\Phi(d), \end{aligned}$$

and for the second term of (4.29) we conclude

$$P\{S(T) > L\} = P\left\{ln\frac{S(T)}{S(0)} > ln\frac{L}{S(0)}\right\}$$
$$\vdots$$
$$= \Phi\big(d - \sigma\sqrt{T}\big). \qquad \square$$

Remark 4.2. In deriving the Black–Scholes PDE (cf. (4.18) and (4.19)), our use of Π of (4.7) as a *continuously changing riskless hedging portfolio* via (4.9) and (4.10) amounts to saying that, to begin with, we took

$$\Pi = \frac{\partial V}{\partial S} S - V \tag{4.30}$$

to be *the value of the portfolio* at time $t \in [0, T]$, and then assumed also that Π satisfies the condition

$$d\Pi = \frac{\partial V}{\partial S} dS - dV. \tag{4.31}$$

This, in turn, leads to the Black–Scholes PDE (cf. (4.18) and (4.19)). Consequently we can conclude that, on assuming (4.31), the Black–Scholes PDE, and thus also the Black–Scholes European call option valuation formula (4.22), can be obtained via the risk–free portfolio approach.

On the other hand, Musiela and Rutkowski (1998, Proposition 5.2.1) show that if Π is given by (4.30) with the function $V(t, S(t)) = C_t(S(t), L, T - t)$ as in (4.22) that solves the Black–Scholes PDE, then the condition (4.31) fails to hold. As a consequence, they conclude that the portfolio Π as in (4.30) is *not self–financing* (cf. Section 5.1.1 in Musiela and Rutkowski, 1998), for if it were, then (4.31) would be satisfied. However, the expected value of the additional cost associated with Π up to time T is zero with respect to the measure generated by $S(t)$ as in (4.25). This property explains why the risk–free portfolio approach leads to the correct form of the Black–Scholes formula. For details substantiating these remarks we refer to Musiela and Rutkowski (1998, Section 5.2), who mention also that Bergman (1982) was most likely first noting that the risk–free portfolio does not satisfy the formal definition of a self–financing strategy. For the latter notion we refer to Section 5.1.1 of Musiela and Rutkowski (1998), who in their Section 5.2 also stress the point that the risk–free portfolio approach is unquestionable in a discrete–time setting. Merton (1973) derived the Black–Scholes equation via constructing a self–financing portfolio containing assets, options and riskless bonds. $\square$

We note in passing that with $S(t)$ as in (4.25) we have

$$\begin{aligned} &E\big(e^{-r(T-t)}S(T)\big|S(t)\big) \\ &= S(t)E\Big(e^{-r(T-t)}\frac{S(T)}{S(t)}\Big|S(t)\Big) \\ &= S(t), \ \ 0 \le t \le T, \end{aligned} \tag{4.32}$$

i.e., the *discounted risk–free asset price process* of the Black–Scholes model

$$e^{-r(T-t)}S(T), \quad 0 \le t \le T, \tag{4.33}$$

is a *martingale* that, in turn, yields also (4.23) as *an alternative view* of the Black–Scholes formula.

Moreover, the valuation procedure of (4.23) in terms of (4.24) ≡ (4.25) can be extended to any European *contingent claim (option)* that is *attainable* (cf., e.g., Section 5.1.2 of Musiela and Rutkowski, 1998) in the arbitrage–free Black–Scholes model, and whose values depend only on the terminal value of the stock price.

We go back to (4.2) for a moment with a standard Wiener process W on some probability space $(\Omega, \mathcal{A}, P)$ and filtration $\mathcal{F}_t$, $0 \le t \le T$, by the values of $\{W(s),\ s \le t\}$ and completed by the addition of sets of P–probability zero.

Briefly, in general, a *European contingent claim* (an *option*) which settles at time T is defined as any $\mathcal{F}_T$–measurable random variable. The *market* is said to be *arbitrage–free* if there exists a probability measure P^*, that is equivalent to P, under which the process $e^{-rt}S(t)$ is a martingale. The *market* is said to be *complete* if and only if this probability P^* is unique. An application of Girsanov's theorem shows that *the arbitrage–free Black–Scholes market is complete* (cf. Lemma 5.1.2 in Musiela and Rutkowski, 1998), *and* that *under* P^*, the *stock price follows the stochastic differential equation of* (4.24), where W *is a new standard* Wiener *process* (though we keep the same notation as in (4.2) for simplicity) under P^*. Also, for the sake of simplicity, we continue writing P instead of P^*, and continue using the same notation $\mathcal{F}_t$ for filtration as well. In this model, let $\pi_t(S(t), T-t)$ stand for the *market value* of, a *fair price* for, a European contingent claim option which settles at time T, and whose values depend only on the terminal value of the stock price. As a generalization of Remark 4.1 we have

Remark 4.3. (cf. Corollary 5.1.3 in Musiela and Rutkowski, 1998). Solving the Black–Scholes PDE (4.19) for $V(t, S(t)) \in C^{1,2}[[0,T) \times (0,\infty)]$, subject to the terminal condition

$$V(T, S(T)) = g(S(T)), \tag{4.34}$$

is equivalent to

$$\begin{aligned} V(t, S(t)) &= e^{-r(T-t)}E(g(S(T))\big|\mathcal{F}_t) \\ &=: \ \pi_t(S(t), T-t), \end{aligned} \tag{4.35}$$

where $g(S(T))$ is the value of an attainable European contingent claim (option) which settles at time T in the arbitrage–free Black–Scholes market whose unique stock price process $S(t)$ is the solution of stochastic differential equation (4.24), and $g(S(T))$ is assumed to be integrable under P of the latter process. □

Remark 4.4. On taking $g(S(T)) = (S(T) - L)^+$ in (4.34), we obtain Remark 4.1. The latter relates Conclusion 4.1, i.e., the solution (4.22) of the Black–Scholes PDE (4.19) subject to (4.20) and (4.21) to the Feynman–Kac type formula (4.23), whose validity we illustrated by direct calculations (cf. (4.28)–(4.29)) of the Black–Scholes formula as stated in (4.26). Roughly speaking, the Feynman–Kac formula expresses the solution of a parabolic PDE as the expected value of a certain functional of a Brownian motion. Indeed, one of the basic ingredients of the proof of Corollary 5.1.3 in Musiela and Rutkowski (1998) (cf. the preceding Remark 4.3, which is a generalization of Remark 4.1) is to show that V as in (4.35) satisfies a special case of the Feynman–Kac PDE (cf. Lemma 5.1.3 in Musiela and Rutkowski, 1998) and this, in turn, leads to concluding also that it satisfies the Black–Scholes PDE (4.19) as well.

As to how to discover V as in (4.35) is based on the general martingale theory we alluded to in our lines right after Remark 4.2 that, in turn, led us to Remark 4.3. Namely, in *the arbitrage–free Black–Scholes complete market*, we have the so–called *risk–neutral valuation formula* which, in terms of P^* as right before Remark 4.3, reads as follows (cf. Corollary 5.1.1 in Musiela and Rutkowski, 1998): Let X be a P^*–attainable European contingent claim which settles at time T. Then the arbitrage price $\pi_t(X)$ at time $t \in [0, T]$ in the arbitrage–free Black–Scholes market is given by the *risk–neutral valuation formula*

$$\pi_t(X) = e^{-r(T-t)} E_{P^*}(X|\mathcal{F}_t), \quad t \in [0, T]. \tag{4.36}$$

In particular, the price of X at time $t = 0$ equals $\pi_0(X) = e^{-rT} E_{P^*}(X)$.

In lieu of explaining new notions, notations and technologies for which we refer to Section 5.1.2 of Musiela and Rutkowski (1998), we "identify" them via adaptation in terms of the already introduced ingredients of Remark 4.3 as follows: X plays the role of $g(S(T))$, P^* is equivalent to P, and $\pi_t(X)$ plays the role of $\pi_t(S(t), T - t)$. □

The papers Black and Scholes (1973), and Merton (1973) were written before the development of the martingale approach to financial mathematics. In fact they were the ones that inspired the latter "fair games" approach to the problems in hand. Our way of arguing Conclusion 4.1 and Remark 4.1, and then "explaining" the latter via Remarks 4.2–4.4 reflects only a mere outline of a small part of this development.

5. On the Black–Scholes fair price formula and volatility

Merton (1973) notes that the manifest characteristic of the Black–Scholes formula (4.26) is the number of variables that it does *not* depend on. Most of all, it does not depend on μ, the expected return of the common stock. Consequently, the risk preferences of the investors for the stock on which the option is based do not affect the option price formula. Inspired by Black and Scholes (1973), this notion of risk neutrality has become an important theme in option pricing theories of economics.

The valuation formula (4.22) depends on five parameters: $S(t)$, $T-t$, L, r and σ. All but the *volatility* $\sigma > 0$ are observable parameters. Hence the usefulness and the hopefully reliable potential applications of the Black–Scholes formula hinge on the investors ability to make reliable forecasts of this volatility parameter of stock prices. *Producing* good, *reliable estimators for* σ for future use *is*, however, *not* an *easy* task. For some details and references on this problem we refer to Section 6.3 in Musiela–Rutkowski (1998).

Roughly speaking, there are two main lines of approach to *estimating* the *volatility* parameter σ that is assumed to be a constant in the stock price process of the Black–Scholes model (cf. (4.2)). Namely, (a) one can use historical data, or (b) one can infer the volatility that is implied via the observed market price of an option.

As to (a) *historical volatility*, the use of historical data to estimate future stock price volatility is usually an unsatisfactory procedure, for stock volatility tends to be unstable through time. □

Concerning (b) *implied volatility*, this procedure is to infer the investment community's concensus on the volatility of a given stock via examining the prices at which options that are based on that stock trade. In particular, the unknown volatility σ of a stock on which one has a European call option is inferred (derived) from its Black–Scholes fair price formula (4.22) $\equiv$ (4.23) via solving the latter at any given time $t \in [0, T]$ for the only unknown parameter σ in it, given that for C_t we take the current market price of this call option. Since C_t is a non–linear equation, it cannot be solved explicitly in terms of $S(t)$, L, r and $T-t$. That C_t is an increasing function of σ helps to create iterative algorithms. Solving for σ along these lines, in "market practice" several implied volatility values are obtained simultaneously from trading values of different European options that are based on the same underlying stock, and a "properly" weighted average of these standard deviations is computed and viewed as the *implied volatility* of the given stock. This information, in turn, can then be used by traders to "set" the volatility value accordingly for the given stock of all the European options that are based on it, and they are interested in. In particular, this means that they would be quoting any such option's market price in terms of the *above arrived at implied volatility.* Thus, in essence, implied volatility becomes a means of quoting option prices. Namely, *given the implied volatility value*, the Black–Scholes fair price value formula (4.22) is now used directly to calculate these "fair" market values that become the quoted European option prices of the traders at time $t \in [0, T]$, when $T-t$ is the time left to maturity time T. Now, if *implied volatility* at any time $t \in [0, T]$, say $\sigma_{\text{imp}}(t)$, were the same for all European call option market prices that are based on the same underlying stock, that is to say, if it were not a function of the contractual features of an option that are parametrized by the remaining time $T-t$ to maturity T and the value L of the strike price, then replacing σ in C_t by *such an implied volatility*, say σ_{imp}, the *thus estimated* C_t of (4.22) would continue to be a fair price formula at any time $t \in [0, T]$ for all European call options that are based on the same stock. Unfortunately, taking for C_t the current market price of a European call option and then solving for σ

the formula of C_t, does not result in producing such a σ_{imp} solution that would possess these desired properties. Nevertheless, the overall concensus in the financial literature seems to be that, in terms of their predictive power, σ_{imp} estimates of volatility outperform more straightforward historical volatility estimates. $\square$

The problem of estimating the volatility σ of various stock prices by σ_{imp} for the sake of the applicability of the Black–Scholes valuation formula C_t (cf. (4.22) $\equiv$ (4.23)) in which σ is the *only* parameter that is not directly observable in the market is (appears to be), at the first sight, somewhat similar to Einstein (1905)'s theory of estimating the Avogadro number N via the use of the postulated diffusion constant $c > 0$ for a Brownian motion (cf. Einstein's assumptions in the paragraph right above (2.1), and (2.1), (2.2), (2.3)) with

$$c = \frac{4kT}{f}, \tag{5.1}$$

where T is the absolute temperature, f is the friction coefficient of the substance in hand, and k is the Boltzman constant. In particular, in liquids e.g., for spherical colloidal particles of radius r we have

$$f = 6\pi\eta r N, \tag{5.2}$$

where η is the viscosity coefficient and N is Avogadro's number. This relationship for f is only valid for particles of such size which obey Stokes' resistance law. Thus, in c of (5.1) with f as in (5.2) all the parameters are *given* under ideal conditions, except Avogadro's number N. Multiplying both sides of (5.1) by $t > 0$, does not change any of the parameters involved. Using his theory of Brownian motion that he *derived via* his basic *assumptions*, Einstein (1905) succeeded in estimating ct for a fixed t, and hence also c. Thus Einstein determined empirically the Avogadro number N. This, in turn, had successfully settled the Avogadro hypothesis (cf. the paragraph containing (2.3)).

Now, in a similar vein, if on taking for C_t its current market price, σ in C_t of (4.22) $\equiv$ (4.23) would not become a function of its other parameters that are all observable, then on replacing it by σ_{imp}, the latter would acquire an "Avogadro number role" in that it would render the thus *estimated* C_t to continue to be a valid value formula for all European call options that are based on the same stock. That this could not have turned out to be true should be, in retrospect, not too surprising in view of the fact that, in combination with (1.1), the Black–Scholes model and formula is based on *assuming*, i.e., not on *deriving*, that stock prices follow geometric Brownian motion (cf. (4.2) $\equiv$ (4.5)) with *constant* coefficients. Though, via (4.11), *randomness* in terms of stochastic differentials *is removed (together with* the otherwise possible *risk preferences* of the investors) from the portfolio (cf. (4.13)) and then, via (4.14), one arrives at Conclusion 4.1 with the option pricing formula C_t as in (4.22), the latter being the solution of the Feynman–Kac type PDE (4.19) (subject to (4.20) and (4.21)), it (i.e., C_t) is necessarily of the Feynman–Kac type form as in (4.23) that of course is very much a function of the terminal condition (4.20) that is based on the strike price L and the geometric Brownian motion in

(4.25) (cf. (4.28)), which in turn, in the Black–Scholes model, is consequential to that of *assuming* (4.5) *to begin with.*

In a nutshell, *after* estimating, by whatever means, the volatility parameter σ of the geometric Brownian motion in (4.25), say by $\hat{\sigma}$, a random variable based on observations on $S(\cdot)$ up to time $t > 0$, the resulting discounted stock price process

$$\hat{S}(t) = \hat{S}(0)e^{(r-\frac{1}{2}\hat{\sigma}^2)t+\hat{\sigma}W(t)} \tag{5.3}$$

ceases to be Gaussian, it will not be the solution of the Itô process of (4.24) any more, nor will

$$\hat{C}_t := S(\hat{S}(t), L, T-t) = V(t, \hat{S}(t)) \tag{5.4}$$

be the solution of the Black–Scholes PDE (4.19) subject to the estimated terminal condition

$$\hat{V} := V(T, \hat{S}(T)) = (\hat{S}(T) - L)^+. \tag{5.5}$$

In particular, letting $\hat{\sigma} = \sigma_{\text{imp}}$, one would not know how $C_{\text{imp}} := C_t$ as in (4.22) with σ_{imp} replacing σ in it, could possibly result from considerations along "the lines of" (5.3)–(5.5). They are mentioned here to highlight the *mathematical difficulties* that we are to face already when having the *statistical problem* of estimating volatility in the simplest possible situation of dealing with only one bank account as in (1.1) and only one Itô process as in (1.4) under the Black–Scholes model as concluded in (4.22) $\equiv$ (4.23).

In spite of the somewhat impossible situation created by difficulties in estimating volatility, the Black–Scholes option pricing formula apparently remains very popular on the trading floor. And, indeed, the very reason for its popularity there seems to be that the only "culprit" in the formula that is not directly observable in the market is volatility. This, in turn, gives an options trader the straight and simple "insight": sell European style derivatives (options) when volatility is high, and buy them when 'volatility' is low. Apparently, option prices obtained within the Black–Scholes framework for short–maturity options are reasonably near to those observed on the option exchanges. As to what extent this coincidence may very well be a consequence of the notoriety of the Black–Scholes formula among traders and other market practitioners remains somewhat of a puzzling question. For example, what if we were to *assume* as an alternative that, instead of the stock price process equation (4.25), we would *start* with a *geometric fractional* Brownian *motion* of the form

$$S_H(t) = S_H(0)\exp\{(r - H\sigma^2)t^{2H} + \sqrt{2H}\sigma W_H(t)\}, \tag{5.6}$$

where the centered Gaussian process $\{W_H(t);\ 0 \le t < \infty\}$ with stationary increments and $W_H(0) = 0$ is a *fractional* Brownian *motion* of order $0 < H < 1$, i.e.,

$$E\big(W_H(t) - W_H(s)\big)^2 = |t-s|^{2H}, \tag{5.7}$$

and

$$W_H(t) - W_H(s) \stackrel{\mathcal{D}}{=} N(0, |t-s|^{2H}) \quad \text{for } \ t, s \geq 0. \tag{5.8}$$

It is of interest to note that $W_{1/2}$ is a standard Wiener process, and $S_{1/2}$ of (5.6) is the discounted stock price process of (4.25). Hence it is of interest to learn if the geometric fractional Brownian motion $\{S_H(t);\ 0 \leq t < \infty,\ 0 < H < 1\}$ as in (5.6) could be regarded as the solution of an appropriate stochastic differential equation driven by W_H. Be it as it may, for the time being, we compute the formula

$$\begin{aligned} C_{0,H}\big(S_H(0), L, T\big) &:= e^{-rT^{2H}} E\big(S_H(T) - L\big)^+ \ \Big| \ \mathcal{F}_0^H\big) \\ &= e^{-rT^{2H}} E\big((S_H(T) - L)^+\big) \\ &= S_H(0)\Phi(d_H) - Le^{-rT^{2H}} \Phi(d_H - \sqrt{2H}\sigma T^H), \end{aligned} \tag{5.9}$$

where

$$d_H = \frac{ln \frac{S_H(0)}{L} + (r + H\sigma^2)T^{2H}}{\sqrt{2H}\sigma T^H}, \tag{5.10}$$

which in turn could possibly be argued as an *alternative* to (4.26) in the Black–Scholes market for $H \neq 1/2$, on noting that $C_{0,1/2} = C_0$ of (4.26). In this regard we should however also point out that now, in addition to the problem of estimating the volatility $\sigma > 0$, we would also have to estimate $H \in (0,1)$, the so-called Hurst parameter. The case of $H \in (1/2, 1)$ is of special interest in view of the thus introduced long range dependence into the stock price process equation (5.6) for S_H.

6. Strong asymptotic properties of integral functionals of geometric stochastic processes

The intensive concentration of many papers on various geometric stochastic processes in financial mathematics has triggered a renewed interest in studying the intrinsic fine analytic properties of such processes. In this section we summarize some of the recent results of E. Csáki–M. Csörgő–A. Földes–P. Révész [CsCsFR] (1998) along these lines. In our paper we study some asymptotic properties of two types of integral functionals of geometric stochastic processes, namely those of

$$A(t) := \int_0^t \exp(X(u))du, \qquad 0 < t < \infty, \tag{6.1}$$

and

$$B(t) := \int_0^\infty \exp\left(V(u) - \frac{u^\alpha}{t}\right) du, \qquad 0 < t < \infty, \tag{6.2}$$

with some $\alpha > 0$. These types of stochastic processes have been extensively investigated in financial mathematics. For example, they yield various option pricings, annuities, etc., by appropriate selection of the processes $X(\cdot)$ and $V(\cdot)$ (cf., e.g., Dufresne, 1989, 1990; Yor, 1992a,b; De Schepper and Goovaerts, 1992; De Schepper, Goovaerts and Delbaen, 1992; Gruet and Shi, 1995; Rogers and Shi, 1995; Goovaerts and Dhaene, 1997). In particular, our investigations were inspired by Gruet and Shi (1995), where they establish integral tests for upper class functions in the special cases

$$A(t) = \int_0^t \exp(W(u))du, \qquad 0 < t < \infty,$$

and

$$B(t) = \int_0^\infty \exp\left(W(u) - \frac{u}{t}\right) du, \qquad 0 < t < \infty,$$

where $\{W(u);\ 0 \le u < \infty\}$ is a standard Brownian motion (Wiener process).

We show that, under fairly general conditions on the respective processes $X(\cdot)$ and $V(\cdot)$, $\log A(t)$ and $\log B(t)$ behave like $\sup_{0\le u\le t} X(u)$ and $\sup_{0\le u<\infty}\big(V(u) - u^\alpha/t\big)$, respectively, for large t. Namely, we establish two strong invariance principles as follows.

Theorem 6.1 (CsCsFR, 1998). *Let the stochastic process $\{X(t); 0 \le t < \infty\}$ have almost surely continuous sample paths, $P(X(0) = 0) = 1$ and put*

$$Z(t) = \log A(t) \quad \text{and} \quad U(t) = \sup_{0\le u\le t} X(u).$$

Assume that for the increments of $X(t)$ we have

$$\sup_{0\le s\le t-a_t}\ \sup_{0\le v\le a_t} |X(s+v) - X(s)| = O\big(r(t,a_t)\big) \quad a.s.$$

as $t \to \infty$, with some nondecreasing a_t $(1 \le a_t \le t)$ and rate $r(t, a_t)$.
Then

$$|Z(t) - U(t)| = O\big(r(t,a_t) + \log t\big) \quad a.s.$$

as $t \to \infty$.

Theorem 6.2 (CsCsFR, 1998). *Let the stochastic process $\{V(t); 0 \le t < \infty\}$ have almost surely continuous sample paths,*

$$P(V(0) = 0) = 1 \quad \text{and} \quad P\left(\limsup_{t\to\infty} V(t) = +\infty\right) = 1.$$

Put

$$Y(t) = \log B(t) \quad \text{and} \quad R(t) = \sup_{0\le u<\infty}\left(V(u) - \frac{u^\alpha}{t}\right).$$

Assume that for the increments of $V(t)$ *we have*

$$\sup_{0\le s\le t-a_t}\ \sup_{0<v\le a_t}\big|V(s+v)-V(s)\big| = O\big(q(t,a_t)\big) \qquad a.s.$$

as $t\to\infty$, *with some nondecreasing* a_t *and rate* $q(t,a_t)$. *Furthermore, suppose that*

$$V(s) < s^{\beta} \quad for \quad s > s_1(\omega), \quad where \ \ 0<\beta<\alpha.$$

Then

$$|Y(t)-R(t)| = O\left(q\left(t^{\frac{1}{\alpha-\beta}}, a_t\right) + \log t\right) \qquad a.s.$$

as $t\to\infty$.

For the process $\{R(t);\ 0<t<\infty\}$ of Theorem 6.2, we also prove quite a general LIL–type result that may be of independent interest on its own, and it reads as follows.

Theorem 6.3 (CsCsFR, 1998). *Let* $V(\cdot)$ *and* $R(\cdot)$ *be as in Theorem* 6.2 *and assume that*

$$\limsup_{t\to\infty}\frac{V(t)}{g(t)} = 1 \ a.s.,$$

where $g(t)>0$ *is regularly varying at infinity with index* ρ $(0<\rho<\alpha)$ *and* $u^{\alpha}/g(u)=h(u)$ *is continuous and strictly increasing for* $u\ge u_0$. *Then*

$$\limsup_{t\to\infty}\frac{tR(t)}{f^{\alpha}(t)} = \left(\frac{\rho}{\alpha}\right)^{\frac{\rho}{\alpha-\rho}} - \left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{\alpha-\rho}} \qquad a.s.,$$

where $f(\cdot)$ *is the inverse of* $h(\cdot)$.

As an application of Theorem 6.1, we *now detail, as in* CsCsFR (1998), *the case of geometric* Brownian *motion* in the Black–Scholes model (cf. (4.25)), where we now put $c = r-\frac{1}{2}\sigma^2$ and let $S(0)=1$. Accordingly, we consider

$$A(t) := \int_0^t \exp(cu+\sigma W(u))du, \tag{6.3}$$

the stock price process that corresponds to that of the *call on average (Asian) option* with $t_0=0$ in the Black–Scholes model.

Studying the asymptotic properties of this stock price process in terms of that of

$$U(t) := \sup_{0\le u\le t}(cu+\sigma W(u)) \tag{6.4}$$

may be of some special interest, for in contrast to the Black–Scholes formula for a European call option, there is not yet a closed analytic fair price formula available for the Asian option. In this regard we refer to L.C.G. Rogers and Z. Shi (1995). Formulating their more general results in terms of our terminologies, they develop the martingale

$$M_t := E\left(\left(\frac{1}{T}A(T) - L\right)^+ \Big| \mathcal{F}_t\right), \quad 0 \le t \le T,$$

and reduce the problem of calculating this value at time $t \in [0, T]$ of an Asian call option with maturity time T and fixed strike price L to solving a Feynman–Kac type parabolic PDE. They provide numerical solutions for their PDE, as well as a lower bound for

$$e^{-rT} E\left(\frac{1}{T}A(T) - L\right)^+,$$

which turns out to be so accurate that, for all practical purposes, it is essentially the true price. For analytical tools which, in turn, lead to quasi–explicit pricing formulas for Asian options, we refer to Geman and Yor (1993).

As to comparing $A(t)$ to $U(t)$, in Theorem 6.1 we select $a_t = 1$. Then we have (cf. M. Csörgő–P. Révész, 1979a, and 1981, Theorem 1.2.1) $r(t, 1) = O\left(\sqrt{\log t}\right)$. Consequently, as $t \to \infty$,

$$\big|Z(t) - U(t)\big| = O(\log t) \quad \text{a.s.,} \tag{6.5}$$

where $Z(t) = \log A(t)$ with $A(t)$ as in (6.3) and $U(t)$ is as in (6.4).

In case $c = 0$, some well-known results can be applied for $U(t)$, yielding the corresponding results for $Z(t)$, some of which are already known. For simplicity suppose $\sigma = 1$. By (6.5) we have a law of the iterated logarithm for $Z(t)$ as follows

$$\limsup_{t\to\infty} \frac{\log \int_0^t \exp(W(u))du}{\sqrt{2t \log\log t}} = 1 \quad \text{a.s.} \tag{6.6}$$

via that of $U(t)$, and, more generally, for the increments of $Z(t)$ with $0 < a_T \le T$ being a monotonically increasing function such that T/a_T is increasing, we obtain

$$\limsup_{T\to\infty} \frac{\sup\limits_{0\le t\le T-a_T} \log \frac{\int_0^{t+a_T} \exp(W(u))du}{\int_0^t \exp(W(u))du}}{\left(a_T\left(\log\frac{T}{a_T} + 2\log\log T\right)\right)^{1/2}} = 1 \quad \text{a.s.,}$$

provided that $a_T(\log T)^{-1} \to \infty$ (cf. Csáki *et al.* (1983)). Moreover, if $\log(T/a_T)(\log\log T)^{-1} \to \infty$, then in (4.5) the limsup can be replaced by lim.

Similarly, for $\log \int_0^t \exp(W(u))du$ we have the same Strassen's LIL, as well as upper and lower class results, as for $\sup_{0\le u\le t} W(u)$. The upper class results were first established by Gruet and Shi

(1995), who gave a direct proof (cf. also Bertoin and Werner (1994), and Keprta (1997)). Moreover, we have as well

$$\lim_{t\to\infty} P\Big(\log \int_0^t \exp(W(u))du < x\sqrt{t}\Big) = \lim_{t\to\infty} P\Big(\sup_{0\le u\le t} W(u) < x\sqrt{t}\Big) = 2\Phi(x) - 1.$$

In case $c < 0$, $\lim_{t\to\infty} A(t)$ is finite almost surely. Hence its stochastic fluctuations will mainly be interesting *in the case of* $c \to 0$, which *basically yields the process* $B(t)$ *with* $\alpha = 1$ in this Wiener case.

As to the case of $c > 0$, it is well-known (cf. Karatzas and Shreve (1988), p. 265) that

$$P\Big(\sup_{s\le t}(\sigma W(s) + cs) \le y\Big) = \Phi\Big(\frac{y - ct}{\sigma\sqrt{t}}\Big) - \exp\Big(\frac{2cy}{\sigma^2}\Big)\Phi\Big(\frac{-y - ct}{\sigma\sqrt{t}}\Big). \tag{6.7}$$

From (6.7) it is easy to conclude also

$$\lim_{t\to\infty} P\Big(\sup_{s\le t}(\sigma W(s) + cs) - ct < y\sqrt{t}\Big) = \Phi\Big(\frac{y}{\sigma}\Big)$$

which, in turn, yields

$$\lim_{t\to\infty} P\Big(\log \int_0^t \exp(\sigma W(u) + cu)du - ct < y\sqrt{t}\Big) = \Phi\Big(\frac{y}{\sigma}\Big).$$

Moreover, observing that

$$ct + \sigma W(t) \le \sup_{0\le u\le t} (cu + \sigma W(u)) \le ct + \sigma \sup_{0\le u\le t} W(u),$$

we arrive at

$$\limsup_{t\to\infty} \frac{\sup_{0\le u\le t} (cu + \sigma W(u)) - ct}{(2\sigma^2 t \log\log t)^{1/2}} = 1 \quad \text{a.s.}$$

Hence, just like in (6.6), we conclude

$$\limsup_{t\to\infty} \frac{\log \int_0^t \exp(\sigma W(u) + cu)du - ct}{(2\sigma^2 t \log\log t)^{1/2}} = 1 \quad \text{a.s.}$$

In order to study the increments of our processes we point out the following simple observation: *for any continuous process* $\eta(t)$ we have

$$\sup_{s\le t+a} \eta(s) - \sup_{s\le t} \eta(s) \le \sup_{v\le a}(\eta(t+v) - \eta(t)). \tag{6.8}$$

Hence by (6.8) we get

$$\sup_{0\le t\le T-a_T} \Big(\sup_{0\le u\le t+a_T} (cu + \sigma W(u)) - \sup_{0\le u\le t} (cu + \sigma W(u))\Big)$$

$$\le ca_T + \sup_{0\le t\le T-a_T} \sup_{0\le v\le a_T} \sigma(W(t+v) - W(t)).$$

Applying (6.5) again, and using Csörgő and Révész (1981, Theorem 1.2.1) again, we arrive at

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \log \frac{\int_0^{t+a_T} \exp(cu+\sigma W(u))du}{\int_0^t \exp(cu+\sigma W(u))du} - ca_T}{\left(2\sigma^2 a_T\left(\log\frac{T}{a_T} + \log\log T\right)\right)^{1/2}} \le 1 \quad \text{a.s.}$$

where $0 \le a_T \le T$ is a nondecreasing function of T for which T/a_T is nondecreasing, and $a_T(\log T)^{-1} \to +\infty$.

In CsCsFR (1998), *we also establish similar results for* $A(t)$, respectively defined in terms of geometric fractional Brownian motion, geometric Gaussian processes in general and Itô processes in particular. Moreover, *we also study the geometric process* $B(t)$ in terms of all these processes, i.e., with V of (6.2) respectively being a fractional Brownian motion, a Gaussian process in general, and an Itô process in particular. Here we present only the case when $V = W$. Namely we now let

$$B(t) = \int_0^\infty \exp\left(W(u) - \frac{u^\alpha}{t}\right) du,$$

where $W(u)$ is a standard Wiener process and $\alpha > 1/2$. By Theorem 1.2.1 of Csörgő and Révész (1981), choosing $a_t = 1$, we have

$$|Y(t) - R(t)| = O(\log t) \qquad a.s.$$

as $t \to \infty$. Recall that here $Y(t) = \log B(t)$ and

$$R(t) = \sup_{0<u<\infty}\left(W(u) - \frac{u^\alpha}{t}\right).$$

We apply Theorem 6.3 with $g(t) = (2t\log\log t)^{1/2}$, $\rho = 1/2$. Since

$$f(t) \sim t^{\frac{2}{2\alpha-1}}(2\log\log t)^{\frac{\alpha}{2\alpha-1}},$$

we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{Y(t)}{t^{\frac{1}{2\alpha-1}}(2\log\log t)^{\frac{\alpha}{2\alpha-1}}} &= \limsup_{t\to\infty}\frac{R(t)}{t^{\frac{1}{2\alpha-1}}(2\log\log t)^{\frac{\alpha}{2\alpha-1}}}\\ &= \left(\frac{1}{2\alpha}\right)^{\frac{1}{2\alpha-1}} - \left(\frac{1}{2\alpha}\right)^{\frac{2\alpha}{2\alpha-1}} \qquad \text{a.s.}\end{aligned}$$

For $\alpha = 1$ we get

$$\limsup_{t\to\infty}\frac{Y(t)}{t\log\log t} = \limsup_{t\to\infty}\frac{R(t)}{t\log\log t} = \frac{1}{2} \qquad \text{a.s.,}$$

and note again that upper and lower classes for $Y(t)$ in this case were obtained by Gruet and Shi (1995) via direct calculations (cf. also Keprta (1997)).

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