A note on self-exciting point processes

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Abstract

The self-exciting property for point processes on the half-line was defined by Kwieciński and Szekli (1996) for each one of three partial orders. Here we unify these definitions of the self-exciting property through a single treatment of an arbitrary closed partial order on the space of integer measures. The self-exciting property is characterized by the behaviour of the compensator of the point process over a class (specific to the order) of "echelon" sets. Our definition recovers and generalizes those of Kwieciński and Szekli.

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Running Head: A note on self-exciting point processes

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1 Introduction

This note is inspired by an article of Kwieciński and Szekli (1996). Their article formalized the self-exciting property of point processes, and established positive association properties for point processes they identified as self-exciting. For three natural partial orders on the space of integer measures on $\mathbf{R}_+$, they defined the self-exciting property in terms of the compensator, intensity and failure rate associated to a point process.

There were certain limitations inherent in Kwieciński and Szekli's definitions. A separate definition of the self-exciting property was required for each partial order; the definitions required that the process have absolutely continuous conditional interarrival distributions; finally, one definition appeared somewhat counter-intuitive, as it regarded renewal processes on the half-line with strictly increasing failure rates as self-exciting with respect to the corresponding order.

In this note we propose a unified approach to defining the self-exciting property based upon testing the compensator measure on a class of order-specific "echelon sets". We thus obtain one specific class of echelon sets for each of the three orders introduced by Kwieciński and Szekli. This approach simplifies the definition of the self-exciting property. Our definition coincides with those of Kwieciński and Szekli in two of three cases, and offers a more intuitive characterization of the self-exciting property in the third case (by which renewal processes with strictly increasing failure rates are not self-exciting). An advantage of our approach is that a (simple) point process is not required to have an intensity to be self-exciting: only the compensator in its regenerative form — which always exists — is invoked. More importantly, the technique may be applied to an arbitrary closed partial order.

Section 2 provides the definitions and main theorems, Section 3 focuses on the self-exciting property as it relates to the ordering $\prec_\infty$, and Section 4 comprises the proofs of results appearing in Section 2.

2 Notation, definitions and main result

Let $\mathbf{N}_0$ denote $\mathbf{N} \cup \{0\}$ and let $\mathcal{N}$ be the space of all measures μ on $\mathbf{R}_+ = [0, \infty)$ such that $\mu(B) \in \mathbf{N}_0$ for all bounded Borel sets $B \subset \mathbf{R}_+$. $\mathcal{N}$ has been shown to be complete and separable under the metric of vague convergence (Grandell (1977)). Let $\mathcal{B}(\mathcal{N})$ denote the Borel σ-algebra generated by this metric. For any $n \in \mathbf{N}_0$ let

$$\tau_n(\mu) := \inf\{t \in \mathbf{R}_+ : \mu([0,t]) \geq n\},$$

where $\inf \emptyset \equiv \infty$ by convention. The measurability of the sets

$$\mathcal{N}_0 := \{\mu \in \mathcal{N} : \forall\, i \in \mathbf{N}, \tau_{i-1}(\mu) < \infty \Rightarrow \tau_{i-1}(\mu) < \tau_i(\mu)\}$$

and

$$\mathcal{N}_0^n := \{\mu \in \mathcal{N}_0 : \tau_n(\mu) < \infty\}.$$

is easily established. For our purposes, a "point process" will always refer to a *simple* point process on $\mathbf{R}_+$, which will denote a random element $N : (\Omega, \mathcal{F}, P) \longrightarrow (\mathcal{N}, \mathcal{B}(\mathcal{N}))$ taking all of its values in $\mathcal{N}_0$. For any $t_1, ..., t_n$ such that $0 < t_1 < t_2 < ... < t_n < \infty$, let

$$F_{n+1}(\cdot; t_1, ..., t_n) := P[\tau_{n+1}(N) - \tau_n(N) \leq \cdot \;|\tau_1(N) = t_1, ..., \tau_n(N) = t_n]$$

be a regular version of the conditional distribution of $\tau_{n+1}(N) - \tau_n(N)$ given $\tau_1(N) = t_1, ..., \tau_n(N) = t_n$. Furthermore, define $\Lambda : \mathbf{R}_+ \times \mathcal{N}_0 \longrightarrow \mathbf{R}_+$ as

$$\Lambda(t, \mu) := \sum_{i=0}^{\infty} \int_{t \wedge \tau_i(\mu)}^{t \wedge \tau_{i+1}(\mu)} \frac{dF_{i+1}(x - \tau_i(\mu); \tau_1(\mu), ..., \tau_i(\mu))}{1 - F_{i+1}((x - \tau_i(\mu))-; \tau_1(\mu), ..., \tau_i(\mu))}.$$

It is a well-known fact that $\Lambda(\cdot, N) : \mathbf{R}_+ \times\, \Omega \longrightarrow \mathbf{R}_+$ represents a predictable version of the compensator of N adapted to N's internal history (Jacod (1975)). When, in addition, the function $F_{n+1}(\cdot; t_1, ..., t_n)$ is absolutely continuous $\forall\, n \in \mathbf{N}_0$ and for all $t_1, ..., t_n$ such that $0 < t_1 < ... < t_n < \infty$, it has a density $f_{n+1}(\cdot; t_1, ..., t_n)$ and a conditional failure rate

$$r_{n+1}(\cdot; t_1, ..., t_n) := \frac{f_{n+1}(\cdot; t_1, ..., t_n)}{1 - F_{n+1}(\cdot; t_1, ..., t_n)}$$

of $\tau_{n+1}(N) - \tau_n(N)$ given $\tau_1(N) = t_1, ..., \tau_n(N) = t_n$ (Kwieciński and Szekli (1996)). The corresponding cumulative hazard function $R_{n+1}(\cdot; t_1, ..., t_n) : F_{n+1}^{-1}(\cdot; t_1, ..., t_n)([0, 1)) \longrightarrow \mathbf{R}_+$ defined by

$$R_{n+1}(t; t_1, ..., t_n) := -\ln(1 - F_{n+1}(t; t_1, ..., t_n)) = \int_0^t r_{n+1}(x; t_1, ..., t_n) dx.$$

also exists. Moreover, under such conditions, $\Lambda(\cdot,\mu)$ itself is, $\forall\ \mu \in \mathcal{N}_0$, absolutely continuous, and has a density $\lambda(\cdot,\mu)$ which satisfies (Kwieciński and Szekli (1996))

$$\lambda(t,\mu) = \sum_{i=0}^{\infty} r_{i+1}(t-\tau_i(\mu);\tau_1(\mu),...\tau_i(\mu))\mathbf{1}_{(\tau_i(\mu),\tau_{i+1}(\mu)]}(t)$$

for all $t \in \mathbf{R}_+$. It should be noted that $\lambda(\cdot,N) : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}_+ \cup \{\infty\}$ represents a predictable version of N's intensity adapted to N's internal history (Brémaud (1981), Theorem III.7).

Kwieciński and Szekli (1996) consider three partial orders on $(\mathcal{N},\mathcal{B}(\mathcal{N}))$ in their definition of the self-exciting property, namely $\prec_{\mathcal{N}}, \prec_{\mathcal{D}}$ and $\prec_{\infty}$. Let $\mu,\nu \in \mathcal{N}$. They write:

- $\mu \prec_{\mathcal{N}} \nu$ if $\mu(B) \leq \nu(B)$ for all Borel sets $B \subseteq \mathbf{R}_+$;
- $\mu \prec_{\mathcal{D}} \nu$ if $\mu((0,t]) \leq \nu((0,t])$ for all $t \in \mathbf{R}_+$;
- $\mu \prec_{\infty} \nu$ if, $\forall\ n \in \mathbf{N}$ such that $\tau_{n-1}(\mu) < \infty$, we have $\tau_{n-1}(\nu) < \infty$ and $\tau_n(\nu) - \tau_{n-1}(\nu) \leq \tau_n(\mu) - \tau_{n-1}(\mu)$.

An important property of these three partial orders is that they are closed, i.e. if $\prec\ \in \{\prec_{\mathcal{N}}, \prec_{\mathcal{D}}, \prec_{\infty}\}$ and $\mu_m \to \mu$, $\nu_m \to \nu$ (vaguely) in $\mathcal{N}$ while $\mu_m \prec \nu_m\ \forall\ m \in \mathbf{N}$, then $\mu \prec \nu$. The closure of these orders follows in a straightforward fashion once it is noted that, if $\mu_m \to \mu$ vaguely, then $\tau_n(\mu_m) \to \tau_n(\nu_m)$ for all $n \in \mathbf{N}_0$.

We are now poised to introduce Kwieciński and Szekli's definition of the self-exciting property.

Definition 2.1 *Let $N : (\Omega,\mathcal{F},P) \longrightarrow (\mathcal{N},\mathcal{B}(\mathcal{N}))$ be a point process on $\mathbf{R}_+$ such that, $\forall\ n \in \mathbf{N}$, $t_1,...,t_{n-1} \in \mathbf{R}_+$ with $0 < t_1 < ... < t_{n-1}$, $F_n(\cdot;t_1,...,t_{n-1})$ is absolutely continuous. Following Kwieciński and Szekli:*

1. *N is self-exciting with respect to $\prec_{\mathcal{N}}$ if, $\forall\ \mu,\ \nu \in \mathcal{N}_0$ such that $\mu \prec_{\mathcal{N}} \nu$, there exist versions of $\lambda(\cdot,\mu)$, $\lambda(\cdot,\nu)$ such that $\lambda(\cdot,\mu) \leq \lambda(\cdot,\nu)$;*
2. *N is self-exciting with respect to $\prec_{\mathcal{D}}$ if $\Lambda(\cdot,\mu) \leq \Lambda(\cdot,\nu)\ \forall\ \mu,\ \nu \in \mathcal{N}_0$ such that $\mu \prec_{\mathcal{D}} \nu$;*

3. *N is self-exciting with respect to $\prec_\infty$ if, $\forall\ \mu,\ \nu \in \mathcal{N}_0$, $\mu \prec_\infty \nu$ implies that $R_n(t;\tau_1(\mu),...,\tau_{n-1}(\mu)) \le R_n(t;\tau_1(\nu),...,\tau_{n-1}(\nu))\ \forall\ t \in \mathbf{R}_+$, $n \in \mathbf{N}$.*

For the sake of brevity we shall henceforward write "KS-self-exciting" instead of "self-exciting...in the sense of Kwieciński and Szekli." Stating the alternate, more general definition of the self-exciting property will require that a new concept be introduced. For any set $A \subseteq \mathbf{R}_+ \times \mathcal{N}$, let $A^\mu := \{t\ \in\ \mathbf{R}_+ :\ (t,\mu) \in A\}$.

Definition 2.2 *A subset $A_{n,x,y}$ of $\mathbf{R}_+ \times \mathcal{N}$, where $n \in \mathbf{N}_0$ and $0 \le x < y < \infty$, is called an echelon set if, $\forall\ \mu \in \mathcal{N}$,*

$$A^\mu_{n,x,y} = \{t\ \in\ \mathbf{R}_+ :\ (t,\mu) \in A_{n,x,y}\} = (\tau_n(\mu) + x, \tau_n(\mu) + y].$$

Echelon sets are notationally convenient tools used in comparing measures over intervals of the form $A^\mu_{n,x,y} = (\tau_n(\mu)+x, \tau_n(\mu)+y]$. The $\mathcal{B}(\mathbf{R}_+)\times\mathcal{B}(\mathcal{N})$-measurability of $A_{n,x,y}$ itself will never be invoked here, and is therefore irrelevant. Echelon sets we shall henceforward mention are ones which, for some closed partial order $\prec$, satisfy the property of "$\prec$-concordance:"

Definition 2.3 *If $\prec$ is a closed partial order on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$, an echelon set $A_{n,x,y}$ is said to be $\prec$-concordant if, $\forall\ \mu,\ \nu \in \mathcal{N}_0^n$ such that $\mu \prec \nu$,*

$$\mu(A^\mu_{n,x,y}) \le \nu(A^\nu_{n,x,y}).$$

Given a closed partial order $\prec$ on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$ and a measure $\mu \in \mathcal{N}_0^n$, the value $\mu(A^\mu_{n,x,y})$ of the μ-measure of the section (at μ) of the $\prec$-concordant set $A_{n,x,y}$ constitutes an "indicator" of how "big" μ is in terms of $\prec$. The self-exciting property may now be defined in general terms:

Definition 2.4 *Let $\prec$ be a closed partial order on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$. A point process $N : (\Omega, \mathcal{F}, P) \longrightarrow (\mathcal{N}, \mathcal{B}(\mathcal{N}))$ is said to be self-exciting with respect to $\prec$ if, for any $\prec$-concordant echelon set $A_{n,x,y} \subseteq \mathbf{R}_+ \times \mathcal{N}$,*

$$\int_{A^\mu_{n,x,y}} \Lambda(dt,\mu) \le \int_{A^\nu_{n,x,y}} \Lambda(dt,\nu)$$

holds whenever $\mu,\ \nu \in \mathcal{N}_0^n$ satisfy $\mu \prec \nu$.

From an intuitive standpoint, Definition 2.4 says that a point process is self-exciting with respect to $\prec$ if its dual predictable projection $\Lambda(\cdot, N)$, conditioned on the past of N being equal to the restriction of μ to that past, will reflect N's tendency to be "as big as μ" in the immediate future by "measuring" $A^{\mu}_{n,x,y}$ in accordance with the size of μ.

Remark 2.5 *The fact that the Lebesgue measure of a section $A^{\mu}_{n,x,y}$ of $A_{n,x,y}$ is always equal to $y - x$ when $\mu \in \mathcal{N}_0^n$, enables us to assert that under Definition 2.4, a Poisson process of constant rate on $\mathbf{R}_+$ is, by the linearity of its compensator measure, self-exciting with respect to all closed partial orders on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$.*

A useful characterization of our definition of the self-exciting property is obtained in the cases $\prec\, = \,\prec_{\mathcal{N}}$, $\prec\, = \,\prec_{\mathcal{D}}$ and $\prec\, = \,\prec_{\infty}$.

Theorem 2.6 *Let $N : (\Omega, \mathcal{F}, P) \longrightarrow (\mathcal{N}, \mathcal{B}(\mathcal{N}))$ be a point process on $\mathbf{R}_+$. Then:*

1. *N is self-exciting with respect to $\prec_{\mathcal{N}}$ in our sense iff*

$$\int_x^y \Lambda\left(dt, \mu\right) \leq \int_x^y \Lambda\left(dt, \nu\right)$$

holds whenever $x < y$ in $\mathbf{R}_+$, $\mu \prec_{\mathcal{N}} \nu$ in $\mathcal{N}_0$;

2. *N is self-exciting with respect to $\prec_{\mathcal{D}}$ in our sense iff $\Lambda(\cdot, \mu) \leq \Lambda(\cdot, \nu)$ for all μ, ν in $\mathcal{N}_0$ such that $\mu \prec_{\mathcal{D}} \nu$.*

3. *N is self-exciting with respect to $\prec_{\infty}$ in our sense iff for all $n \in \mathbf{N}_0$, $y > 0$ and $\mu, \nu \in \mathcal{N}_0^n$ with $\mu \prec_{\infty} \nu$ we have*

$$\int_{\tau_n(\mu)}^{\tau_n(\mu)+y} \Lambda(dt, \mu) \leq \int_{\tau_n(\nu)}^{\tau_n(\nu)+y} \Lambda(dt, \nu).$$

A pleasant consequence of Theorem 2.6 is that our definition strongly resembles that of Kwieciński and Szekli in the aforementioned three cases.

Corollary 2.7 *Let $N : (\Omega, \mathcal{F}, P) \longrightarrow (\mathcal{N}, \mathcal{B}(\mathcal{N}))$ be a point process on $\mathbf{R}_+$ such that, for all $n \in \mathbf{N}$, $t_1, ..., t_{n-1} \in \mathbf{R}_+$ with $0 < t_1 < ... < t_{n-1}$, $F_n(\cdot; t_1, ..., t_{n-1})$ is absolutely continuous. Then:*

1. *N is KS-self-exciting with respect to $\prec_{\mathcal{N}}$ iff N is self-exciting with respect to $\prec_{\mathcal{N}}$ in our sense;*

2. *N is KS-self-exciting with respect to $\prec_{\mathcal{D}}$ iff N is self-exciting with respect to $\prec_{\mathcal{D}}$ in our sense;*

3. *If N is self-exciting with respect to $\prec_{\infty}$ in our sense, then N is KS-self-exciting with respect to $\prec_{\infty}$.*

The proofs of Theorem 2.6 and Corollary 2.7 constitute Section 4. Note that, as explained in Section 3, the converse of part (3) of Corollary 2.7 does not hold. The reader should also be warned that, although $\mu \prec_{\mathcal{N}} \nu$ or $\mu \prec_{\infty} \nu$ implies $\mu \prec_{\mathcal{D}} \nu$, counterexamples exist to show that, when $\prec_1$, $\prec_2$ $\in \{\prec_{\mathcal{N}}, \prec_{\mathcal{D}}, \prec_{\infty}\}$ and $\prec_1 \neq \prec_2$, N being self-exciting with respect to $\prec_1$ *does not* imply that N is self-exciting with respect to $\prec_2$ — and this, even in our sense. Corollary 2.7 permits us to use the counterexamples of Kwieciński and Szekli when $\prec_1 \in \{\prec_{\mathcal{N}}, \prec_{\mathcal{D}}\}$ (see the paragraph following Definition 4.1 of Kwieciński and Szekli (1996)). Finally, Example 3.2 of the present note shows that a point process may be self-exciting with respect to $\prec_{\infty}$ in our sense without being self-exciting with respect to either $\prec_{\mathcal{N}}$ or $\prec_{\mathcal{D}}$.

Kwieciński and Szekli (1996) established the positive association of processes they identified as self-exciting with respect to one of the three aforementioned closed partial orders. We conjecture an extension, subject to some restrictions, of this result to point processes we define as self-exciting with respect to an arbitrary closed partial order (Plante (1998)).

3 About the case $\prec = \prec_{\infty}$

One may ask whether part (3) of Corollary 2.7 has a converse, i.e., is a point process KS-self-exciting with respect to $\prec_{\infty}$ necessarily self-exciting with respect to $\prec_{\infty}$ in our sense? The answer is negative. It is known, for example, that all renewal processes on $\mathbf{R}_+$ which have absolutely continuous interarrival distribution functions are, due to the mutual independence of their interarrival times, KS-self-exciting with respect to $\prec_{\infty}$ (see the remark following Lemma 4.2 of Kwieciński and Szekli (1996)). They are not, however, necessarily self-exciting with respect to $\prec_{\infty}$ in our sense. As a matter of fact, the following theorem, similar to Lemma 4.2 of Kwieciński and

Szekli (1996) on renewal processes that are self-exciting with respect to $\prec_{\mathcal{D}}$, states that renewal processes which are self-exciting with respect to $\prec_\infty$ in our sense, and satisfy certain regularity conditions, are time-homogeneous Poisson processes.

Theorem 3.1 *Let $N : (\Omega, \mathcal{F}, P) \longrightarrow (\mathcal{N}, \mathcal{B}(\mathcal{N}))$ be a renewal process on $\mathbf{R}_+$ with an absolutely continuous lifetime distribution, and with a failure rate $r : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ whose domain may be partitioned into a sequence $\{I_n\}_{n=0}^\infty$ of intervals $I_n := [a_n, a_{n+1})$ with $a_{n+1} > a_n$, and such that, $\forall\ n \geq 0$, r is strictly increasing, constant, or strictly decreasing over I_n. Then, N is self-exciting with respect to $\prec_\infty$ in our sense iff N is a Poisson process of constant rate r.*

<u>**Proof:**</u> Part ($\Longleftarrow$) follows from Remark 2.5. Part ($\Longrightarrow$) will be achieved in three steps. Assume that N is self-exciting with respect to $\prec_\infty$ in our sense. Since r is of bounded variation by hypothesis, we may, without loss of generality, assume that r is right-continuous.

<u>Step 1</u>: r is constant over $I_0 = [a_0, a_1) = [0, a_1)$.

Suppose that r is strictly increasing over I_0. Let $\mu \prec_\infty \nu \in \mathcal{N}_0$ be such that $\tau_1(\mu) = a_1$, $\tau_1(\nu) = a_1/2$, and $\tau_2(\nu) = a_1$. Then,

$$\int_{\tau_0(\mu)}^{\tau_0(\mu)+a_1} \Lambda(du, \mu) = \int_0^{a_1} r(u)du > 2\int_0^{a_1/2} r(u)du = \int_{\tau_0(\nu)}^{\tau_0(\nu)+a_1} \Lambda(du, \nu),$$

contradicting the hypothesis that N is self-exciting with respect to $\prec_\infty$. If r is strictly decreasing over I_0, let μ, $\{\nu_n\}_{n\in\mathbf{N}}$ in $\mathcal{N}_0$ be such that $\mu \prec_\infty \nu_n$ for all $n \in \mathbf{N}$ such that $2/n < a_1/2$, that $\tau_1(\mu) = a_1/2$, $\tau_2(\mu) = a_1$, and that $\tau_1(\nu_n) = 1/n$, $\tau_2(\nu_n) = 2/n$ and $\tau_3(\nu_n) = a_1\ \forall\ n \in \mathbf{N}$. Then,

$$\int_{\tau_0(\mu)}^{\tau_0(\mu)+a_1} \Lambda(du, \mu) = 2\int_0^{a_1/2} r(u)du$$

$$> \int_0^{a_1} r(u)du = \lim_{n\longrightarrow\infty}(2\int_0^{1/n} r(u)du + \int_0^{a_1-2/n} r(u)du)$$

$$= \lim_{n\longrightarrow\infty} \int_{\tau_0(\nu)}^{\tau_0(\nu)+a_1} \Lambda(du, \nu_n),$$

which contradicts the hypothesis.

Step 2: $r(a_1) = \lim_{s \to a_1^+} r(s) = r(0)$.

Suppose that $r(a_1) > r(0)$. By the right-continuity of r, there exists an interval $I = [a_1, t) \neq \emptyset$ with $I \subset I_1 = [a_1, a_2)$ such that $r(s) > r(0)\ \forall\ s \in I$. Without loss of generality assume that $t - a_1 < a_1 - a_0 = a_1$. Let $\mu \prec_\infty \nu \in \mathcal{N}_0$ be such that $\tau_1(\mu) = t,\ \tau_2(\mu) = a_2 + t$ and that $\tau_1(\nu) = t - a_1,\ \tau_2(\nu) = t$. Then,

$$\int_{\tau_0(\mu)}^{\tau_0(\mu)+t} \Lambda(du, \mu) = \int_0^t r(u)du = \int_0^{a_1} r(u)du + \int_{a_1}^t r(u)du$$

$$> \int_0^{t-a_1} r(u)du + \int_0^{a_1} r(u)du = \int_{\tau_0(\nu)}^{\tau_0(\nu)+t} \Lambda(du, \nu),$$

contradicting the hypothesis.
Now suppose that $r(a_1) < r(0)$. By the right-continuity of r, there exists an interval $I = [a_1, t) \neq \emptyset$ with $I \subset I_1 = [a_1, a_2)$ such that $r(s) < r(0)\ \forall\ s \in I$. Without loss of generality assume that $t - a_1 < a_1 - a_0 = a_1$. Let $\mu,\ \{\nu_n\}_{n \in \mathbf{N}} \in \mathcal{N}_0$ be such that $\mu \prec_\infty \nu_n$ for all $n \in \mathbf{N}$ such that $2/n < t - a_1$, that $\tau_1(\mu) = t - a_1$, $\tau_2(\mu) = t$, and that $\tau_1(\nu_n) = 1/n$, $\tau_2(\nu_n) = 2/n$, $\tau_3(\nu_n) > t$ for all $n \in \mathbf{N}$. We have that

$$\int_{\tau_0(\mu)}^{\tau_0(\mu)+t} \Lambda(du, \mu) = \int_0^{t-a_1} r(u)du + \int_0^{a_1} r(u)du$$

$$> \int_0^t r(u)du = \lim_{n \to \infty} (2 \int_0^{1/n} r(u)du + \int_0^{t-2/n} r(u)du)$$

$$= \lim_{n \to \infty} \int_{\tau_0(\nu_n)}^{\tau_0(\nu_n)+t} \Lambda(du, \mu),$$

contradicting the hypothesis. We conclude that $r(a_1) = r(0)$.

Step 3: r is constant over $I_1 = [a_1, a_2)$ (and therefore over $I_0 \cup I_1$).

Suppose that r is strictly increasing over I_1. There thus exists an interval $I = (a_1, t) \neq \emptyset$ with $I \subset I_1 = [a_1, a_2)$ such that $r(s) > r(0) \forall\ s \in I$. Proceed as in the first argument of Step 2 to obtain a contradiction.

Similarly, suppose that r is strictly decreasing over $I_1 = [a_1, a_2)$. There thus exists an interval $I = (a_1, t) \neq \emptyset$ with $I \subset I_1 = [a_1, a_2)$ such that $r(s) < r(0) \, \forall \, s \in I$. Proceed as in the second argument of Step 2 to infer a contradiction. r is therefore constant over I_1.

Steps 1, 2 and 3 establish that r is constant over $I_0 \cup I_1$. Moreover, iterating Steps 2 and 3 n times will yield that r is constant over $I_0 \cup I_1 \cup ... \cup I_n$ for all $n \in \mathbf{N}$. Thus, the hypothesis implies that r is constant over $\mathbf{R}_+ = \cup_{n \in \mathbf{N}} I_n$, entailing that $\lambda(\cdot, N) : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is constant with $\lambda(\cdot, N) \equiv r$, which implies, at last, that N is a Poisson process of rate r by Watanabe's theorem (Brémaud (1981), Theorem II.5).

Q.E.D.

Although the class of point processes which are self-exciting with respect to $\prec_\infty$ appears, in our sense, narrower than in the sense of Kwieciński and Szekli, one may also argue that our definition is more intuitive. Indeed, it is difficult to construe renewal processes with strictly increasing failure rates as being self-exciting with respect to any ordering. Kwieciński and Szekli's definition of the self-exciting property affirms that such processes are self-exciting with respect to $\prec_\infty$, while ours does not.

We conclude this section with an example that our class of point processes self-exciting with respect to $\prec_\infty$ does not trivially reduce to Poisson processes of constant rate.

Example 3.2

Let $N : (\mathcal{N}_0, \mathcal{B}(\mathcal{N}) \cap \mathcal{N}_0) \longrightarrow (\mathcal{N}, \mathcal{B}(\mathcal{N}))$ be a canonical (identity) point process whose distribution admits an intensity $\lambda(\cdot, N)$ given by

$$\lambda(t, \mu) := \mathbf{I}_{[t \leq \tau_2(\mu)]} + \left[\left(\frac{1}{\tau_2(\mu) - \tau_1(\mu)} \right) \vee 1 \right] \mathbf{I}_{[t > \tau_2(\mu)]}$$

for $\mu \in \mathcal{N}_0$, $t \in \mathbf{R}_+$. Such a point process exists by Theorem 3.6 of Jacod (1975). A simple, but lengthy, argument shows that

$$\lambda(\tau_n(\mu) + t, \mu) \leq \lambda(\tau_n(\nu) + t, \nu)$$

holds for all $n \in \mathbf{N}_0$, $t > 0$ and $\mu \prec_\infty \nu \in \mathcal{N}_0^n$, whence it is clear that N is self-exciting with respect to $\prec_\infty$ in our sense. N is not, however, self-exciting with respect to $\prec_\mathcal{N}$ or $\prec_\mathcal{D}$. Let μ, $\nu \in \mathcal{N}_0$ be such that $\tau_1(\nu) = 1/10$, $\tau_2(\nu) = 3/10$, $\tau_3(\nu) = 2/5$, $\tau_4(\nu) = \infty$; $\tau_1(\mu) = 3/10$, $\tau_2(\mu) = 2/5$, and $\tau_3(\mu) = \infty$. Observe that $\mu \prec_\mathcal{N} \nu$ and $\mu \prec_\mathcal{D} \nu$. N is not self-exciting with respect to $\prec_\mathcal{N}$ because

$$\int_1^2 \Lambda\left(dt, \mu\right) = 10 > 5 = \int_1^2 \Lambda\left(dt, \nu\right),$$

and N is not self-exciting with respect to $\prec_\mathcal{D}$ because

$$\int_0^2 \Lambda\left(dt, \mu\right) = \int_0^{2/5} \lambda\left(t, \mu\right) dt + \int_{2/5}^2 \lambda\left(t, \mu\right) dt = 16.4$$

$$> 8.8 = \int_0^{3/10} \lambda\left(t, \nu\right) dt + \int_{3/10}^2 \lambda\left(t, \nu\right) dt = \int_0^2 \Lambda\left(dt, \nu\right).$$

4 Proofs of Theorem 2.6 and Corollary 2.7

A preliminary observation is required characterizing $\prec$-concordant echelon sets in the cases $\prec = \prec_\mathcal{N}$, $\prec = \prec_\mathcal{D}$ and $\prec = \prec_\infty$.

Proposition 4.1 *Let $A_{n,x,y}$ be an echelon set. Then:*

1. *$A_{n,x,y}$ is $\prec_\mathcal{N}$-concordant iff $n = 0$;*
2. *$A_{n,x,y}$ is $\prec_\mathcal{D}$-concordant iff $n = 0$ and $x = 0$;*
3. *$A_{n,x,y}$ is $\prec_\infty$-concordant iff $x = 0$.*

Proof

1. If $n = 0$, then $\forall\ \mu$, $\nu \in \mathcal{N}_0^0 = \mathcal{N}_0$ such that $\mu \prec_\mathcal{N} \nu$ we have $\mu(A^\mu_{0,x,y}) = \mu((x, y]) \leq \nu((x, y]) = \nu(A^\nu_{0,x,y})$, which implies that $A_{n,x,y} = A_{0,x,y}$ is $\prec_\mathcal{N}$-concordant.
If $n \neq 0$, let $\nu \in \mathcal{N}_0^n$ be such that $\tau_{n+1}(\nu) = \tau_n(\nu) + y + 1$ and $\tau_i(\nu) = \tau_{i-1}(\nu) + (x + y)/2$ for $i \geq n + 2$. Construct $\mu \in \mathcal{N}_0^n$ such that $\tau_i(\mu) = \tau_i(\nu)$ for $i \in \{1, ..., n-1\}$ and $\tau_i(\mu) = \tau_{i+1}(\nu)$ for $i \geq n$. Then, clearly, $\mu \prec_\mathcal{N} \nu$ but

$\mu(A^\mu_{n,x,y}) > \nu(A^\nu_{n,x,y}) = 0$. Therefore, $A_{n,x,y}$ is not $\prec_{\mathcal{N}}$-concordant if $n \neq 0$.

2. If $n = 0 = x$, then $\forall\ \mu,\ \nu \in \mathcal{N}_0^0 = \mathcal{N}_0$ such that $\mu \prec_{\mathcal{D}} \nu$ we have $\mu(A^\mu_{0,0,y}) = \mu((0,y]) \leq \nu((0,y]) = \nu(A^\nu_{0,0,y})$, which implies that $A_{n,x,y} = A_{0,0,y}$ is $\prec_{\mathcal{D}}$-concordant.
If $n \neq 0$, construct μ and ν as for (1). Then $\mu \prec_{\mathcal{D}} \nu$ but $\mu(A^\mu_{n,x,y}) > \nu(A^\nu_{n,x,y})$, so $A_{n,x,y}$ is not $\prec_{\mathcal{D}}$-concordant if $n \neq 0$.
Now consider $A_{n,x,y} = A_{0,x,y}$ with $x \neq 0$. Let $\nu \in \mathcal{N}_0$ be such that $\tau_1(\nu) = x/2$ and $\tau_2(\nu) = y+1$; let $\mu \in \mathcal{N}_0$ be such that $\tau_1(\mu) = (x+y)/2$ and $\tau_i(\mu) = \tau_i(\nu)\ \forall\ i \geq 2$. $A_{0,x,y}$ is not $\prec_{\mathcal{D}}$-concordant because $\mu \prec_{\mathcal{D}} \nu$, while $\mu(A^\mu_{0,x,y}) > \nu(A^\nu_{0,x,y})$.

3. If $x = 0$, then $\forall\ \mu,\ \nu \in \mathcal{N}_0^n$ such that $\mu \prec_\infty \nu$ we have $\mu(A^\mu_{n,0,y}) = \mu((\tau_n(\mu), \tau_n(\mu)+y]) \leq \nu((\tau_n(\nu), \tau_n(\nu)+y]) = \nu(A^\nu_{n,0,y})$. Therefore, $A_{n,x,y} = A_{n,0,y}$ is $\prec_\infty$-concordant if $x = 0$.
If $x \neq 0$, let $\nu \in \mathcal{N}_0^n$ be such that $\tau_{n+1}(\nu) = [2\tau_n(\nu) + x]/2$, $\tau_{n+2}(\nu) = y+1$, and $\tau_i(\nu) = \tau_{i-1}(\nu)+1$ for $i \geq n+3$; let $\mu \in \mathcal{N}_0$ be such that $\tau_i(\mu) = \tau_i(\nu)$ for $i \in \{1,..,n\}$, $\tau_{n+1}(\mu) = \tau_n(\nu)+(x+y)/2$, and $\tau_i(\mu) = \tau_{i-1}(\mu)+[\tau_i(\nu)-\tau_{i-1}(\nu)]$ for $i \geq n+2$. Then $\mu \prec_\infty \nu$ but $\mu(A^\mu_{n,x,y}) > \nu(A^\nu_{n,x,y})$, whence $A_{n,x,y}$ is not $\prec_\infty$-concordant if $x \neq 0$.

Q.E.D.

<u>Proof of Theorem 2.6</u>

1. This is evident in the light of Definition 2.4 and the fact that $\prec_{\mathcal{N}}$-concordant sets are of the form $A_{n,x,y} = A_{0,x,y}$.

2. N is self-exciting with respect to $\prec_{\mathcal{D}}$ in our sense iff, $\forall\ y > 0$, and for all μ, ν in $\mathcal{N}_0$ such that $\mu \prec_{\mathcal{D}} \nu$,

$$\int_{A^\mu_{0,0,y}} \Lambda(dt,\mu) = \int_0^y \Lambda(dt,\mu) = \Lambda(y,\mu)$$

$$\leq \Lambda(y,\nu) = \int_0^y \Lambda(dt,\nu) = \int_{A^\nu_{0,0,y}} \Lambda(dt,\nu).$$

(2) follows.

3. This is obvious if one considers Definition 2.4 along with the fact that $\prec_\infty$-concordant sets are of the form $A_{n,x,y} = A_{n,0,y}$.

Q.E.D.

Proof of Corollary 2.7

Recall that absolute continuity of the $F_n(\cdot, t_1, ..., t_{n-1})$ is assumed.

1. Suppose that N is KS-self-exciting, i.e. that $\forall\ \mu,\ \nu \in \mathcal{N}_0$ such that $\mu \prec_{\mathcal{N}} \nu$, there exist versions of $\lambda(\cdot,\mu)$, $\lambda(\cdot,\nu)$ such that $\lambda(\cdot,\mu) \leq \lambda(\cdot,\nu)$. Then, for any $\prec$-concordant echelon set $A_{n,x,y} = A_{0,x,y}$ and for any $\mu \prec_{\mathcal{N}} \nu$ in $\mathcal{N}_0$ we obtain

$$\int_{A^{\mu}_{0,x,y}} \Lambda(dt,\mu) = \int_x^y \lambda(t,\mu)dt \leq \int_x^y \lambda(t,\nu)dt = \int_{A^{\nu}_{0,x,y}} \Lambda(dt,\nu),$$

whence N is self-exciting with respect to $\prec_{\mathcal{N}}$ in our sense.
Suppose now that N is self-exciting with respect to $\prec_{\mathcal{N}}$ in our sense. Pick μ, $\nu \in \mathcal{N}_0$ such that $\mu \prec_{\mathcal{N}} \nu$. Since $\Lambda(\cdot,\mu)$ and $\Lambda(\cdot,\nu)$ are absolutely continuous, they are differentiable almost everywhere with respect to the Lebesque measure. At a point t of common differentiability the hypothesis that N is self-exciting with respect to $\prec_{\mathcal{N}}$ in our sense entails

$$\Lambda'(t,\mu) = \lim_{s\to t^+} \frac{\Lambda(s,\mu) - \Lambda(t,\mu)}{s-t} = \lim_{s\to t^+} \frac{\int_{A^{\mu}_{0,s,t}} \Lambda(du,\mu)}{s-t}$$

$$\leq \lim_{s\to t^+} \frac{\int_{A^{\nu}_{0,s,t}} \Lambda(du,\nu)}{s-t} = \lim_{s\to t^+} \frac{\Lambda(s,\nu) - \Lambda(t,\nu)}{s-t} = \Lambda'(t,\nu).$$

Since $\lambda(\cdot,\mu)$ and $\lambda(\cdot,\mu)$ act as densities of $\Lambda(\cdot,\mu)$ and $\Lambda(\cdot,\nu)$ respectively, a classic result of real analysis implies that $\lambda(\cdot,\mu) = \Lambda'(\cdot,\mu)$ and $\lambda(\cdot,\nu) = \Lambda'(\cdot,\nu)$ almost everywhere. Thus, there exist versions of the functions $f_n(\cdot;t_1,...,t_n)$ which ensure that $\lambda(t,\mu) \leq \lambda(t,\nu)\ \forall\ t \in \mathbf{R}_+$.

2. This is an immediate consequence of part (2) of Theorem 2.6.

3. Suppose that N is self-exciting with respect to $\prec_\infty$ in our sense. We must show that, $\forall\, n \in \mathbf{N}$, $y \in \mathbf{R}_+$,

$$R_{n+1}(y; \tau_1(\mu), ..., \tau_n(\mu)) \leq R_{n+1}(y; \tau_1(\nu), ..., \tau_n(\nu))$$

whenever $\mu \prec_\infty \nu$ in $\mathcal{N}_0$. Note that if $\tau_n(\mu) = \infty$ or $\tau_n(\nu) = \infty$, we may attribute an arbitrary value to one side of the inequality. Suppose, therefore, that $\tau_n(\mu)$, $\tau_n(\nu) < \infty$, i.e., that μ, $\nu \in \mathcal{N}_0^n$. Let μ', $\nu' \in \mathcal{N}_0^n$ be such that $\mu' \prec_\infty \nu'$, that $\tau_i(\mu') = \tau_i(\mu)$, $\tau_i(\nu') = \tau_i(\nu)$ $\forall\, i \in \{1, ..., n\}$, and that $\tau_{n+1}(\mu')$, $\tau_{n+1}(\nu') > y$. By hypothesis, for any $\prec_\infty$-concordant set $A_{n,x,y} = A_{n,0,y}$, we have

$$\int_{A^{\mu'}_{n,0,y}} \Lambda(du, \mu') = \int_{\tau_n(\mu')}^{\tau_n(\mu')+y} \lambda(u, \mu')du$$

$$= \int_0^y r_{n+1}(u; \tau_1(\mu'), ..., \tau_n(\mu'))du$$

$$= R_{n+1}(y; \tau_1(\mu'), ..., \tau_n(\mu'))$$

$$\leq R_{n+1}(y; \tau_1(\nu'), ..., \tau_n(\nu'))$$

$$= \int_0^y r_{n+1}(u; \tau_1(\nu'), ..., \tau_n(\nu'))du$$

$$= \int_{\tau_n(\nu')}^{\tau_n(\nu')+y} \lambda(u, \nu')du = \int_{A^{\nu'}_{n,0,y}} \Lambda(du, \nu').$$

The conclusion follows from the fact that

$$R_{n+1}(y; \tau_1(\mu), ..., \tau_n(\mu)) = R_{n+1}(y; \tau_1(\mu'), ..., \tau_n(\mu'))$$

and

$$R_{n+1}(y; \tau_1(\nu), ..., \tau_n(\nu)) = R_{n+1}(y; \tau_1(\nu'), ..., \tau_n(\nu')).$$

Q.E.D.

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