Testing Multiple Changes in the Mean

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Abstract

In this paper we investigate testing the null hypothesis H_0 of no-change in the mean of independent observations against the alternative $H_A^{(s)}$ of at most s changes in the mean. We construct an s-parameter stochastic process and study the distribution of its sup-functional to test H_0 against $H_A^{(s)}$. As a special case, the epidemic alternative is investigated.

1 Introduction

We are to test the *no-change in the mean* null-hypothesis

> H_0 : $X_1, \dots, X_n$ are independent identically distributed random variables with $\mathbb{E}X_i = \mu$ and $0 < \sigma^2 = VarX_i < \infty$, $1 \le i \le n$,

against the *at most s changes in the mean* ($s \in \mathbf{N}, 1 \le s < n$) alternative

> $H_A^{(s)}$: $X_1, \dots, X_n$ are independent random variables and there are s integers $\tau_1 = \tau_1(n), \tau_2 = \tau_2(n), \dots, \tau_s = \tau_s(n)$, $1 \le \tau_1 \le \tau_2 \le \dots \le \tau_s < n$, such that $\mu_1 = \mathbb{E}X_1 = \dots = \mathbb{E}X_{\tau_1} \ne \mu_2 = \mathbb{E}X_{\tau_1+1} = \dots = \mathbb{E}X_{\tau_2}$, $\mu_2 = \mathbb{E}X_{\tau_1+1} = \dots = \mathbb{E}X_{\tau_2} \ne \mu_3 = \mathbb{E}X_{\tau_2+1} = \dots = \mathbb{E}X_{\tau_3}$, $\dots$, $\mu_s = \mathbb{E}X_{\tau_{s-1}+1} = \dots = \mathbb{E}X_{\tau_s} \ne \mu_{s+1} = \mathbb{E}X_{\tau_s+1} = \dots = \mathbb{E}X_n$, and $0 < \sigma^2 = VarX_i < \infty$, $1 \le i \le n$.

Change-point problems have been considerably studied in the literature from the parametric as well as the nonparametric point of view. In particular, the problem of at most one (single) change in the mean (i.e., $s = 1$) has been investigated in detail (cf., for example, Brodsky and Darkhovsky (1993) and Csörgő and Horváth (1988a, 1988b, 1997)). In this situation it seems natural to compare the mean of the first k observations (before

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the change) to the mean of the last $n-k$ observations (after the change), i.e., to study functionals based on

$$\frac{1}{k}\sum_{i=1}^{k} X_i - \frac{1}{n-k}\sum_{i=k+1}^{n} X_i \;=\; \frac{n}{k(n-k)}\Big(S(k)-\frac{k}{n}S(n)\Big), \quad 1\le k<n,$$

where $S(k) := \sum_{i=1}^{k} X_i$.

Let us assume throughout this paper that the variance σ^2, $0<\sigma^2<\infty$, is known. Otherwise, given H_0, it can be estimated in a consistent way. One can, for example, use the sample variance, or the pooled variances as suggested by Csörgő and Horváth (1997, p.70). They showed the consistency of the pooled variances in case of $s=1$ and concluded that this estimator is preferable to the sample variance. In case of testing for more than one change, $s>1$, a modified version of the pooled variances also seems to be superior.

Continuing with the latter stochastic process, we base a sequence of statistics on it, namely we define

$$T_n^{(1)} \;:=\; \max_{1\le k<n} \frac{|S(k)-\frac{k}{n}S(n)|}{\frac{k}{n}(1-\frac{k}{n})n^{1/2}\sigma}.$$

However, this sequence converges in probability to ∞, as $n\to\infty$, even if the null hypothesis of having no change in the mean were to be true. This problem may, for example, be avoided by either considering the statistics

$$T_n^{(2)} \;:=\; \max_{1\le k<n} \frac{|S(k)-\frac{k}{n}S(n)|}{n^{1/2}\sigma}$$

or

$$T_n^{(3)} \;:=\; \max_{1\le k<n} \frac{|S(k)-\frac{k}{n}S(n)|}{\Big(\frac{k}{n}(1-\frac{k}{n})\log\log\frac{1}{\frac{k}{n}(1-\frac{k}{n})}\Big)^{1/2} n^{1/2}\sigma}.$$

Under the null hypothesis H_0 Donsker's invariance principle yields, as $n\to\infty$,

$$T_n^{(2)} \;\xrightarrow{\mathcal{D}}\; \sup_{0<t<1}|B(t)|, \tag{1.1}$$

and, on account of Szyszkowicz (1996, 1997) (cf. Theorem 2.1.1 in Csörgő and Horváth (1997)), as $n\to\infty$, we have

$$T_n^{(3)} \;\xrightarrow{\mathcal{D}}\; \sup_{0<t<1}\frac{|B(t)|}{\Big(t(1-t)\log\log\frac{1}{t(1-t)}\Big)^{1/2}},$$

where $\{B(t); 0\le t\le 1\}$ is a Brownian bridge. Rejecting H_0 for large values of $T_n^{(2)}$, respectively for those of $T_n^{(3)}$, both tests are consistent for testing H_0 against $H_A^{(1)}$. Tests based on $T_n^{(2)}$ should be more powerful for detecting changes that occur in the middle, namely near $\frac{n}{2}$, than for the ones occurring near the endpoints 0 and n. On the other hand, tests based on $T_n^{(3)}$ should emphasize possible changes near the endpoints, while retaining sensitivity to possible changes in the middle as well.

Suppose we were to use the statistic $T_n^{(2)}$ to test for more than one change-point, for example, to test H_0 against $H_A^{(2)}$. Immediately the following question arises: 'Is $T_n^{(2)}$ consistent for testing H_0 against $H_A^{(2)}$?'. We shall see in Section 2 that the latter statistic is consistent even when testing H_0 against $H_A^{(s)}$.

However, from a theoretical point of view that may be of interest in studying the performance (power) of the test statistics in hand, it seems to be more natural to construct a 2-parameter stochastic process for the sake of testing for at most 2 changes in the mean. This, in turn, leads to studying the functionals (statistics)

$$T_n^{(4)} := \max_{1\le k_1<k_2<n} \frac{\left|\frac{k_2}{n}S(k_1)+\frac{n-k_1}{n}S(k_2)-\frac{k_2}{n}S(n)\right|}{n^{1/2}\sigma}, \tag{1.2}$$

as we shall do in Section 3.

2 Using $T_n^{(2)}$ to test for multiple changes

The problem of testing for at most one (single) change in the mean via using $T_n^{(2)}$ has been investigated in detail. Its asymptotic behavior under H_0 as well as under $H_A^{(1)}$ is already known (we refer, for example, to Csörgő and Horváth (1988a, 1988b)). However, from a practical point of view, it seems to be natural to investigate its behavior under the alternative $H_A^{(s)}$ as well. As an example, we may think of s, the true number of change-points, being unknown. Having in mind the latter example, for the sake of establishing consistency of rejecting H_0 for large values of $T_n^{(2)}$, it would be desirable to show that under $H_A^{(s)}$ for any integer $s \ge 1$, $T_n^{(2)}$ converges in probability to ∞, as $n \to \infty$.

To do so, we investigate the limiting behavior of $\frac{\sigma}{n^{1/2}}T_n^{(2)}$ and state the following theorem.

Proposition 2.1 *Assume that $H_A^{(s)}$ holds. If $\tau_i = \tau_i(n) := [n\lambda_i]$, $i = 1, 2\ldots, s$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s < 1$, then, as $n \to \infty$,*

$$\frac{S([(n+1)t]) - tS(n)}{n} \xrightarrow{a.s.} \bar{u}_{\lambda_1,\lambda_2,\ldots,\lambda_s}(t), \quad 0 < t < 1,$$

where

$$\bar{u}_{\lambda_1,\lambda_2,\ldots,\lambda_s}(t) := \Big(\sum_{i=0}^{m_t}(\lambda_i - \lambda_{i-1})\mu_i - \lambda_{m_t}\mu_{m_t+1}\Big) - t\Big(\sum_{i=0}^{s+1}(\lambda_i - \lambda_{i-1})\mu_i - \mu_{m_t+1}\Big),$$

with $m_t := \min\{q : \lambda_q < t \le \lambda_{q+1},$ where $0 =: \lambda_0 < \lambda_1 < \cdots < \lambda_s < \lambda_{s+1} := 1\}$, $\lambda_{-1} := 0$ and $\mu_0 := 0$.

To prove this proposition, we note that $\mathbb{E}S([(n+1)t]) = \sum_{i=0}^{m_t}(\tau_i - \tau_{i-1})\mu_i + ([(n+1)t] - \tau_{m_t})\mu_{m_t+1}$ and $\mathbb{E}tS(n) = t\sum_{i=0}^{s+1}(\tau_i - \tau_{i-1})\mu_i$, where $\tau_0 := 1$ and $\tau_{s+1} := n$. Applying Kolmogorov's SLLN we get the desired result.

We note at the outs that m_t takes integer values from 0 to s, hence the limiting function $\bar{u}_{\lambda_1,\lambda_2,\ldots,\lambda_s}(t)$, $0<t<1$, consists of $s+1$ different parts. Furthermore, it is easy to see, that this limiting function is equal to 0 for all $0<t<1$ if and only if $\mu_1=\mu_2=\cdots=\mu_{s+1}$.

Assuming $0<\sigma^2<\infty$, Proposition 2.1 combined with (1.1) implies the consistency of testing H_0 against $H_A^{(s)}$, $s\geq 1$, via large values of $T_n^{(2)}$, namely

$$\mathbb{P}\{H_0 \text{ is rejected when using } T_n^{(2)} | H_A^{(s)} \text{ is true}\} \underset{n\to\infty}{\longrightarrow} 1.$$

A similar test that is based on large values of $T_n^{(3)}$ is also consistent when testing for more than one change-point. Tables for the limiting distribution of $T_n^{(3)}$ under H_0 may be found in Eastwood and Eastwood (1998).

3 A test based on an s-parameter stochastic process

Although we studied in Section 2 the asymptotic behavior of $T_n^{(2)}$ under $H_A^{(s)}$ and showed its consistency to test for multiple changes, this statistic was constructed in particular to test for at most one change-point (compare the mean before to the mean after a possible change). As mentioned earlier in Section 1, it seems to be natural to construct an s-parameter stochastic process when s change-points are expected. We will do so by using a geometrical argument.

For simplicity, let us first assume that we are testing for at most 2 changes. Consider the linear function $m(t):=t$, $t\in\mathbf{R}$, which joins under H_0 all the points $(k,\frac{1}{\mu}\mathbb{E}\{S(k)\})$, $k\in\mathbf{N}$, if $\mu\neq 0$, and it joins all the points $(k,\mathbb{E}\{S(k)\})$, $k\in\mathbf{N}$, if $\mu=0$.

Without loss of generality let us assume that $\mu=1$. Then we join in Figure 1 all the points $(k,\mathbb{E}\{S(k)\}\ )$, $k\in\mathbf{N}$, via the straight line $m(t)=t$. We pick one $k_1\in\{1,\ldots,n-2\}$ and then one $k_2\in\{k_1+1,\ldots,n-1\}$ (We note that in Figure 1 we may think of k_1 and k_2 being defined as $k_1:=[nt_1^{(fix)}]$ and $k_2:=[nt_2^{(fix)}]$ respectively, $0<t_1^{(fix)}<t_2^{(fix)}<1$). We draw a horizontal line starting from $B:=(0,\mathbb{E}\{S(k_1)\})$, containing the point $(k_1,\mathbb{E}\{S(k_1)\})$, and with terminus $C:=(k_2,\mathbb{E}\{S(k_1)\})$. We draw a vertical line from the terminus and intersect the t-axis. We denote this intersection by $D:=(k_2,\mathbb{E}\{S(0)\})$, where we define $S(0):=0$. In this way we construct a rectangle, denoted by $ABCD$ (see Figure 1), where $A:=(0,\mathbb{E}\{S(0)\})$, with length k_2 and height $\mathbb{E}\{S(k_1)\}$. We also draw a horizontal line starting from $F:=(k_1,\mathbb{E}\{S(k_2)\})$, containing the point $(k_2,\mathbb{E}\{S(k_2)\})$, and with terminus $G:=(n,\mathbb{E}\{S(k_2)\})$. We then draw a vertical line from the starting point and another one from the terminus, both intersecting the t-axis at $E:=(k_1,\mathbb{E}\{S(0)\})$ and $H:=(n,\mathbb{E}\{S(0)\})$ respectively. Similarly, we construct another rectangle, denoted by $EFGH$ (see Figure 1), with length $(n-k_1)$ and height $\mathbb{E}\{S(k_2)\}$.

Reflecting each point of the rectangle $EFGH$ around the 45 degree line $m=t$, we get the new rectangle $BIJC$, where $B:=(0,\mathbb{E}\{S(k_1)\})$ is the reflection point of $E:=(k_1,\mathbb{E}\{S(0)\})$, $I:=(0,\mathbb{E}\{S(n)\})$ is the reflection point of $H:=(n,\mathbb{E}\{S(0)\})$, $J:=(k_2,\mathbb{E}\{S(n)\})$ is the reflection point of $G:=(n,\mathbb{E}\{S(k_2)\})$ and $C:=(k_2,\mathbb{E}\{S(k_1)\})$ is the reflection point of $F:=(k_1,\mathbb{E}\{S(k_2)\})$. This new rectangle has length k_2 and height $\mathbb{E}\{S(n)-S(k_1)\}$.

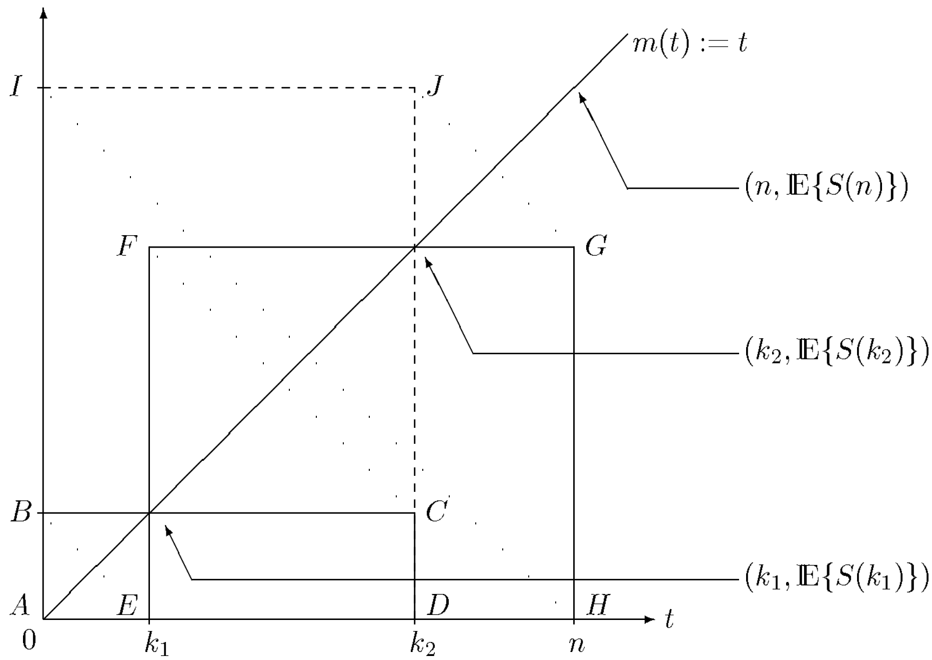

Figure 1: A geometrical interpretation of $\mathbb{E}\{k_2 S(k_1) + (n - k_1)S(k_2) - k_2 S(n)\} = 0$ under H_0.

Combining the two rectangles $ABCD$ and $BIJC$ with each other, we have constructed the rectangle $AIJD$, which has length k_2 and height $\mathbb{E}\{S(n)\}$. Moreover, under H_0, the new rectangle $AIJD$ has the same area as the sum of the two other ones, namely $ABCD$ and $EFGH$. Consequently, we have that

$$k_2 \times \mathbb{E}\{S(k_1)\} + (n - k_1) \times \mathbb{E}\{S(k_2)\} - k_2 \times \mathbb{E}\{S(n)\} \quad = \quad 0.$$

Thus, in principle, for each given k_1 and k_2, $1 \leq k_1 < k_2 < n$, we constructed an unbiased estimator of zero assuming that H_0 is true. We may also say that, viewed this way, testing for at most two changes in the mean results in comparing areas of three different rectangles with each other.

Similarly, when testing for s changes, for each given $k_1, k_2, \ldots, k_s$, $1 \leq k_1 < k_2 < \cdots < k_s < n$, there correspond the s rectangles with endpoints $(k_{i-1}, \mathbb{E}\{S(0)\})$, $(k_{i-1}, \mathbb{E}\{S(k_i)\})$, $(k_{i+1}, \mathbb{E}\{S(k_i)\})$ and $(k_{i+1}, \mathbb{E}\{S(0)\})$, $1 \leq i \leq s$, respectively, where $k_0 := 0$ and $k_{s+1} := n$. Each of these rectangles with length $(k_{i+1} - k_{i-1})$ and height $\mathbb{E}\{S(k_i)\}$ has area $(k_{i+1} - k_{i-1}) \times \mathbb{E}\{S(k_i)\}$, $1 \leq i \leq s$, respectively, where $k_0 := 0$ and $k_{s+1} := n$.

By reflecting appropriate parts of these (overlapping) rectangles around the 45 degree line $m = t$ we can construct a new rectangle with endpoints $(0, \mathbb{E}\{S(0)\})$, $(0, \mathbb{E}\{S(n)\})$, $(k_s, \mathbb{E}\{S(n)\})$ and $(k_s, \mathbb{E}\{S(0)\})$. This rectangle, with length k_s, height $\mathbb{E}\{S(n)\}$ and area $k_s \times \mathbb{E}\{S(n)\}$, has now, under H_0, the same area as the sum of the previous areas.

Consequently, for each given combination of $1 \le k_1 < k_2 < \cdots < k_s < n$,

$$T^{(5)}_{k_1,k_2,\ldots,k_s,n} = \sum_{i=1}^{s}(k_{i+1} - k_{i-1})S(k_i) - k_s S(n),$$

with $k_0 := 0$ and $k_{s+1} := n$, is an unbiased estimator of zero under H_0. Moreover, let us define

$$T^{(5)}_n := \max_{1\le k_1<k_2<\cdots<k_s<n} \frac{1}{n^{3/2}\sigma}\Big|T^{(5)}_{k_1,k_2,\ldots,k_s,n}\Big|. \tag{3.1}$$

Since $T^{(5)}_{k_1,k_2,\ldots,k_s,n}$ is a linear combination of partial sums and $t_i = \frac{k_i}{n}$, $1 \le i \le s$, as a consequence of Csörgő and Révész (1981, Theorem S.2.2.1 by Major (1979) combined with (S.2.2.2)) under H_0, as $n \to \infty$, we have

$$T^{(5)}_n \xrightarrow{\mathcal{D}} \sup_{0<t_1<t_2<\cdots<t_s<1}\Big|\sum_{i=1}^{s}(t_{i+1} - t_{i-1})W(t_i) - t_s W(1)\Big|, \tag{3.2}$$

with $t_0 := 0$ and $t_{s+1} := 1$. Hence, the limiting distribution under H_0 is defined via a linear combination of a standard Wiener process $\{W(t); 0 \le t \le 1\}$. Producing tables for the latter limiting function is desirable for testing H_0 against $H_A^{(s)}$.

To investigate the limiting behavior of $T^{(5)}_n$ under $H_A^{(s)}$ we state the following theorem.

Theorem 3.1 *Assume that $H_A^{(s)}$ holds. Define $t_0 := 0$ and $t_{s+1} := \frac{n}{n+1}$. If $\tau_i = \tau_i(n) := [n\lambda_i]$, $i = 1, 2\ldots, s$, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_s < 1$, then, as $n \to \infty$,*

$$\frac{\sum_{i=1}^{s}(t_{i+1} - t_{i-1})S([(n+1)t_i]) - t_s S(n)}{n} \xrightarrow{a.s.} \tilde{u}_{\lambda_1,\lambda_2,\ldots,\lambda_s}(t_1, t_2, \ldots, t_s),$$

where

$$\begin{aligned}\tilde{u}_{\lambda_1,\lambda_2,\ldots,\lambda_s}(t_1, t_2, \ldots, t_s) &= \sum_{i=1}^{s}(t_{i+1} - t_{i-1})\Big(\sum_{j=0}^{m_{t_i}}(\lambda_j - \lambda_{j-1})\mu_j + (t_i - \lambda_{m_{t_i}})\mu_{m_{t_i}+1}\Big)\\ &\quad - t_s\sum_{i=0}^{s+1}(\lambda_i - \lambda_{i-1})\mu_i, \quad 0 < t_1 < t_2 < \cdots < t_s < 1,\end{aligned}$$

with $m_{t_i} := \min\{q : \lambda_q < t_i \le \lambda_{q+1},$ where $0 =: \lambda_0 < \lambda_1 < \cdots < \lambda_s < \lambda_{s+1} := 1\}$, $1 \le i \le s$, $\lambda_{-1} := 0$ and $\mu_0 := 0$.

The proof is similar to that of Proposition 2.1. We note at the outs that the limiting function $\tilde{u}_{\lambda_1,\lambda_2,\ldots,\lambda_s}(t_1, t_2, \ldots, t_s)$, $0 < t_1 < t_2 < \cdots < t_s < 1$, consists of $s(s+1)$ different parts. Furthermore, it is easy to see, that this limiting function is equal to 0 for all $0 < t_1 < t_2 < \cdots < t_s < 1$ if and only if $\mu_1 = \mu_2 = \cdots = \mu_{s+1}$.

Assuming $0 < \sigma^2 < \infty$, Theorem 3.1 combined with (3.2) implies the consistency of testing H_0 against $H_A^{(s)}$, $s \ge 1$, via large values of $T^{(5)}_n$.

Remark 3.1 *We succeeded in constructing the two statistics $T_n^{(2)}$ and $T_n^{(5)}$ that can be used to test for multiple changes in the mean. However, nothing is known about the power of these two tests. It is expected that in most cases tests based on $T_n^{(5)}$ will perform better than those based on $T_n^{(2)}$ when testing H_0 versus $H_A^{(s)}$. A numerical study is desirable, but to do so, the limiting distribution of $T_n^{(5)}$ for different $s > 1$ under H_0 has to be computed first.*

Remark 3.2 *We note that for $s = 1$ the two statistics $T_n^{(2)}$ and $T_n^{(5)}$ coincide and so do of course their corresponding limiting results under H_0 and $H_A^{(1)}$ as well.*

Remark 3.3 *We also note that for each integer $s \geq 1$ the test based on large values of the statistic $T_n^{(5)}$ is consistent for testing H_0 against $H_A^{(s)}$. Furthermore, in principle, for each s we can compute the corresponding limiting distribution function of $T_n^{(5)}$ under H_0. Hence, under H_0 all tests based on them will reject $H_A^{(s)}$, but under $H_A^{(s)}$ for any fixed s, any of them might consistently pick up more than s change-points just as the test based on $T_n^{(2)}$. Therefore, it is not clear how to go about estimating the true number of change-points via using tests build on $T_n^{(5)}$ for different $s \geq 1$.*

4 The epidemic alternative

In the previous section we studied alternatives with at most s changes in the mean. As a special case, we will test for two changes and will also require that the means before the first and after the second change are the same. This kind of alternative, more or less as formulated by Levin and Kline (1985), has been called the epidemic alternative, on postulating that an epidemic state runs from time τ_1 through τ_2, after which the normal state is restored as it was before time τ_1. Applications of this model in econometric context are studied by Broemeling and Tsurumi (1987).

We consider the case, where we have an epidemic change in the mean as assumed in the alternative $H_A^{(2)}$ below. In particular, we test the *no-change in the mean* null-hypothesis

> H_0 : *$X_1, \ldots, X_n$ are independent identically distributed random variables with $\mathbb{E}X_i = \mu$ and $0 < \sigma^2 = Var X_i < \infty$, $1 \leq i \leq n$,*

against the *epidemic change in the mean* alternative

> $H_A^{(2)}$: *$X_1, \ldots, X_n$ are independent random variables and there are two integers τ_1 and τ_2, $1 \leq \tau_1 < \tau_2 < n$, such that $\mu_1 = \mathbb{E}X_1 = \cdots = \mathbb{E}X_{\tau_1} = \mathbb{E}X_{\tau_2+1} = \cdots = \mathbb{E}X_n$, $\mu_2 = \mathbb{E}X_{\tau_1+1} = \cdots = \mathbb{E}X_{\tau_2}$, $\mathbb{E}X_{\tau_1} \neq \mathbb{E}X_{\tau_1+1}$ and $0 < \sigma^2 = Var X_i < \infty$, $1 \leq i \leq n$.*

Nonparametric tests for epidemic alternatives were discussed in the literature in the past two or so decades (cf. Csörgő and Horváth (1997, Section 2.8.4) and Yao (1993) and their related references).

Using the results from Section 3 we have that under H_0, $T_n^{(5)} := T_n^{(4)}$ of (1.2) converges in distribution to the supremum of a linear combination of a standard Wiener process (cf. (3.2) with $s = 2$) and under $H_A^{(2)}$ it converges in probability to ∞, since $\frac{\sigma}{n^{1/2}} T_n^{(5)}$ converges in probability to the supremum of the limiting function $\tilde{u}_{\lambda_1,\lambda_2}(t_1, t_2)$ defined as

$$\begin{cases} (\lambda_2 - \lambda_1) t_2 (\mu_1 - \mu_2), & 0 < t_1 < t_2 \le \lambda_1 < \lambda_2 < 1, \\ \Big((t_2 - \lambda_1)(-1 + \lambda_2 + t_1) + (\lambda_2 - t_2)\lambda_1 \Big)(\mu_1 - \mu_2), & 0 < t_1 \le \lambda_1 < t_2 \le \lambda_2 < 1, \\ (\lambda_2 - \lambda_1)(-1 + t_2 + t_1)(\mu_1 - \mu_2), & 0 < t_1 \le \lambda_1 < \lambda_2 < t_2 < 1, \\ \Big((\lambda_2 - t_1)\lambda_1 - (1 - \lambda_2)(t_2 - \lambda_1) \Big)(\mu_1 - \mu_2), & 0 < \lambda_1 < t_1 < t_2 \le \lambda_2 < 1, \\ \Big(-(1 - \lambda_2)(t_1 - \lambda_1) + (\lambda_2 - t_1)(-1 + t_2 + \lambda_1) \Big) & \\ \quad \cdot (\mu_1 - \mu_2), & 0 < \lambda_1 < t_1 \le \lambda_2 < t_2 < 1, \\ -(1 - t_1)(\lambda_2 - \lambda_1)(\mu_1 - \mu_2), & 0 < \lambda_1 < \lambda_2 < t_1 < t_2 < 1, \end{cases}$$

where $\tau_1 = [n\lambda_1]$, $\tau_2 = [n\lambda_2]$ and $\mu_1 \neq \mu_2$.

In case of $\mu_1 \neq \mu_2$ the limiting function $\tilde{u}_{\lambda_1,\lambda_2}(t_1, t_2)$, $0 < t_1 < t_2 < 1$, as above takes on negative as well as positive values and neither a maximum nor a minimum is taken at the times of change $(t_1, t_2) = (\lambda_1, \lambda_2)$. Hence, estimating the two change-points via, for example, the argument where $\tilde{u}_{\lambda_1,\lambda_2}(t_1, t_2)$, $0 < t_1 < t_2 < 1$, takes its maximum in t_1 and t_2 will not give us a good estimator in general. This is due to the fact that for $\mu_1 \neq \mu_2$ we have, for example, $\min\{\tilde{u}_{\lambda_1,\lambda_2}(\lambda_1/2, \lambda_1), \tilde{u}_{\lambda_1,\lambda_2}(\lambda_2, (\lambda_2+1)/2)\} < \tilde{u}_{\lambda_1,\lambda_2}(\lambda_1, \lambda_2) < \max\{\tilde{u}_{\lambda_1,\lambda_2}(\lambda_1/2, \lambda_1)\tilde{u}_{\lambda_1,\lambda_2}(\lambda_2, (\lambda_2+1)/2)\}$. Therefore, the problem of estimating the times of at most 2 changes still seems to be an open problem, even when we restrict ourselves to the epidemic alternative.

Conclusion

Using a geometrical argument we constructed a statistic based on an s-parameter stochastic process to test for s, $s \geq 1$, changes in the mean (cf. Section 3) and showed its consistency. Furthermore, we showed the consistency for testing H_0 against $H_A^{(s)}$, $s \geq 1$, when using a statistic which was originally introduced to test for at most one change-point (cf. Section 2). As a special case we also investigated the so-called epidemic alternative (cf. Section 4) and showed that even in this simplified case estimating the change-points is not an easy task.

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References

[1] Brodsky, B.E. and Darkhovsky, B.S. (1993). Nonparametric Methods in Change-Point Problems. Kluwer, Dordrecht.

[2] Broemeling, L.D. and Tsurumi, H. (1987). Econometric and Structural Change. Marcel Dekker. New York.

[3] Csörgő, M. and Horváth, L. (1988a). *Nonparametric Methods for Changepoint Problems.* Handbook of Statistics, Vol. **7** 403-425. Elsevier Science Publishers B.V. (Nort-Holland).

[4] Csörgő, M. and Horváth, L. (1988b). *Invariance Principles for Changepoint Problems.* Journal of Multivariate Analysis **27**, 151-168.

[5] Csörgő, M. and Horváth, L. (1997). Limit Theorems in Change-point Analysis. John Wiley. Chichester.

[6] Csörgő, M. and Révész, P. (1981). Strong Approximations in Probability and Statistics. Academic Press, New York.

[7] Eastwood, B.J. and Eastwood, V.R. (1998). *Tabulating Weighted Functionals of Brownian Bridges via Monte Carlo Simulation.* Asymptotic Methods in Probability and Statistics - a Volume in Honour of Miklós Csörgő (ed. B. Szyszkowicz), 707-719, Elsevier Science B.V., Amsterdam.

[8] Levin, B. and Kline, J. (1985). *The Cusum Test of Homogeneity with an Application in Spontaneous Abortion Epidemiology.* Statist. Med., **4**, 469-488.

[9] Major, P. (1979). *An Improvement of Strassen's Invariance Principle.* Ann. Probability, **7**, 55-61.

[10] Orasch, M. (1999). Multiple Change-points with an Application to Financial Modeling. PhD Thesis, Carleton University, Ottawa, Canada.

[11] Szyszkowicz, B. (1996). *Weighted Approximations of Partial Sum Processes in* $D[0,\infty)$*. I.* Studia Sci. Math. Hungar., **31**, 323-353.

[12] Szyszkowicz, B. (1997). *Weighted Approximations of Partial Sum Processes in* $D[0,\infty)$*. II.* Studia Sci. Math. Hungar., **33**, 305-320.

[13] Yao, Q. (1993). *Tests for Change-points with Epidemic Alternatives.* Biometrika, **80**, 179-191.