PRE–SUPER BROWNIAN MOTION

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Abstract

At time 0 there are N particles located at the origin $0 \in I\!\!R^d$. They execute independent Brownian motions and, at a certain time ψ, they either die or split with probability 1/2. Right after time ψ the new particles continue to execute independent Brownian motions, independently from those of their ancestors, up to time 2ψ, when they again repeat the just formulated process. This process will be called pre–super Brownian motion.

We conclude a time sequence of exact distribution functions for the most–right vertex of the quadrant in $I\!\!R^d$ that is determined by the surviving particles, as well as an asymptotic form of these distributions when $\psi = N^{-1}$, and $N \to \infty$. We also prove a strong theorem when $\psi = 1$ and $N \to \infty$.

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1. Introduction

In this paper we investigate the following
MODEL

(i) N ($N = 1, 2, \ldots$) particles start from the position $0 \in \mathbb{R}^d$ and execute N independent Brownian motions (Wiener processes) $W_1(t), W_2(t), \ldots, W_N(t)$ ($W_i(t) \in \mathbb{R}^d$, $0 \leq t < \infty$, $i = 1, 2, \ldots, N$),

(ii) arriving at time $t = \psi$ to the new locations $W_1(\psi), W_2(\psi), \ldots, W_N(\psi)$, they die,

(iii) at death they are replaced by $Z_1, Z_2, \ldots, Z_N$ offspring (respectively), where $Z_1, Z_2, \ldots, Z_N$ are i.i.d.r.v.'s (also independent from $W_i(t)$ ($i = 1, 2, \ldots, N$)) with

$$\mathbf{P}\{Z_i = 0\} = \mathbf{P}\{Z_i = 2\} = \frac{1}{2},$$

(iv) each offspring, starting from where its ancestor dies, executes a Brownian motion (Wiener process) (from its starting point, between $t = \psi$ and $t = 2\psi$) and repeats the above given steps. Wiener processes and offspring-numbers are assumed to be independent of one another.

Let

(a) $B^*(t, \psi, N)$ ($t = 0, \psi, 2\psi, \ldots$) be the number of particles living at time $t = i\psi$, the particles born at time $i\psi$ to be counted as being alive at time $i\psi$, but not at time $(i+1)\psi$, i.e., to begin with, for $i = 0, 1$, respectively, we have

$$B^*(0, \psi, N) = N,$$

$$\mathbf{P}\{B^*(\psi, \psi, N) = 2k\} = \binom{N}{k} 2^{-N} \quad (k = 0, 1, 2, \ldots, N),$$

(b) $B(t, \psi, N)$ ($t = 0, \psi, 2\psi, \ldots$) be the number of those particles (among the N ancestors) which have at least one living offspring at time t, i.e., to begin with, at times $t = 0, \psi$, respectively, we have

$$B(0, \psi, N) = N,$$

$$\mathbf{P}\{B(\psi, \psi, N) = k\} = \binom{N}{k} 2^{-N} \quad (k = 0, 1, 2, \ldots, N).$$

Clearly, for any $t \geq \psi$ we have

$$0 \leq B(t, \psi, N) \leq N, \quad B^*(t, \psi, N) \geq 2B(t, \psi, N),$$
$$\{B(t, \psi, N) = 0\} = \{B^*(t, \psi, N) = 0\},$$

(c) $X_{t1}, X_{t2}, \ldots, X_{tB^*(t,\psi,N)}$ be the locations of the particles at time t in $\mathbb{R}^d$,

(d) $\lambda(A, t, \psi, N) := \#\{i : 1 \leq i \leq B^*(t, \psi, N),\ X_{ti} \in A\}$, where A is a Borel set of $\mathbb{R}^d$,

(e) $A(x) := \{y \in \mathbb{R}^d : y < x\}$, $x \in \mathbb{R}^d$, where $x < y$ is meant to be a componentwise inequality,

(f) $\lambda(x,t,\psi,N) := \lambda(A(x),t,\psi,N)$,

(g) $F(x,t,\psi,N) := \dfrac{\lambda(xt^{1/2},t,\psi,N)}{B^*(t,\psi,N)}$,

(h) $X_{ti} := (X_{ti}(1), X_{ti}(2), \ldots, X_{ti}(d)), \quad i = 1,\ldots,B^*(t,\psi,N)$,

(i) $M_{t\ell} = \max_{1\le i\le B^*(t,\psi,N)} X_{ti}(\ell), \quad (\ell = 1,2,\ldots,d)$,

(j) $M(t) = M(t,\psi,N) := (M_{t1}, M_{t2}, \ldots, M_{td})$,

(k) $\mathcal{M}_t(x,\psi,N) := \mathbf{P}\{M(t) < xt^{1/2} \mid B^*(t,\psi,N) > 0\}, \quad t = 0,\psi,2\psi,\ldots$.

$M(t)$ will be called the most-right vertex of the quadrant determined by the points X_{ti} $(i = 1,2,\ldots,B^*(t,\psi,N))$ in $\mathbb{R}^d$. We call the sequence $\lambda(A,t,\psi,N)$ $(t = 0,\psi,2\psi,\ldots)$ of random measures a

(i) critical branching Wiener process if $\psi = N = 1$ (cf. [4]),

(ii) pre-super Brownian motion if $\psi = N^{-1}$ $(N = 1,2,\ldots)$ (cf., e.g., [1], [2], [3], for a more detailed description and study of this kind of branching diffusions).

For the sake of simplicity, in case

(i) $\psi = N = 1$, we omit the variables ψ, N, i.e. we write $\lambda(A,t,1,1) = \lambda(A,t)$, $B^*(t,1,1) = B^*(t),\ldots$, etc.,

(ii) $\psi = N^{-1}$, we omit ψ, i.e. we write $\lambda(A,t,N^{-1},N) = \lambda(A,t,N)$, $B^*(t,N^{-1},N) = B^*(t,N),\ldots$, etc.

In Section 2 we collect some known results on the case $\psi = N = 1$. Section 3 is devoted to a study of the case $\psi = N^{-1}$. In Section 4 we prove a strong theorem when $\psi = 1$ and $N \to \infty$.

2. Critical branching Wiener process

Lemma 2.1 ([4]) *For any $t = 0,1,2,\ldots$ we have*

$$\mathbf{E}B^*(t) = 1, \tag{2.1}$$

$$\mathbf{E}(B^*(t))^2 = t+1 \tag{2.2}$$

$$\lim_{t\to\infty} B^*(t) = 0 \quad a.s. \tag{2.3}$$

Lemma 2.2 ([5]) *For any $t = 0,1,2,\ldots$ we have*

$$\frac{2}{t+2+2\log(t+1)} \le p_t := \mathbf{P}\{B^*(t) > 0\} \le \frac{2}{t+2}. \tag{2.4}$$

Theorem 2.1 ([5])

(i) $\mathcal{M}_t(x)$ *converges weakly to a distribution function* $\mathcal{M}(x)$ *as* $t \to \infty$,

(ii) *for any* $0 < R < 1$ *there exists a* $C = C(R, d) > 0$ *such that*

$$\frac{1}{2\pi}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) \le 1 - \mathcal{M}_t(x) \le C\exp\left(-\frac{x^2}{2} + \frac{x^2}{(\log x)^R}\right)$$

for any $t > 0$, $x > 1$,

(iii) *for any* $0 < R < 1$ *there exists a* $C = C(R, d) > 0$ *such that*

$$\frac{1}{2\pi}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) \le 1 - \mathcal{M}(x) \le C\exp\left(-\frac{x^2}{2} + \frac{x^2}{(\log x)^R}\right) \tag{2.5}$$

for any $x > 1$,

(iv) $\mathcal{M}(\cdot)$ *is a solution of the integral equation*

$$F(x) = \int_0^1 \int_{I\!R^d} (F(\alpha^{-1/2}(x-y)))^2 \phi_\alpha(y)dyd\alpha, \tag{2.6}$$

where

$$\phi_\alpha(y) = (2\pi(1-\alpha))^{-d/2}\exp\left(-\frac{y^2}{2(1-\alpha)}\right),$$

with $y^2 := \langle y, y\rangle$, *and* $\mathcal{M}(\cdot)$ *is the only distribution function which satisfies* (2.5) *and* (2.6).

3. Pre-super Brownian motion

In the light of the MODEL (cf. our Introduction) and the results quoted in Section 2, the next two lemmas are immediate. Hence we state them without proof.

Lemma 3.1

$$\mathbf{P}\{B(t, \psi, N) = k\} = \binom{N}{k} p_{t/\psi}^k (1 - p_{t/\psi})^{N-k}$$

$(k = 0, 1, 2, \ldots, N,\ t = \psi, 2\psi, \ldots)$, *where the sequence* $\{p_.\}$ *is defined in* Lemma 2.2.

Lemma 3.2

$$M(t\psi, \psi, 1) \stackrel{\mathcal{D}}{=} \psi^{1/2} M(t, 1, 1),$$
$$\mathcal{M}_{t\psi}(x, \psi, 1) = \mathcal{M}_t(x, 1, 1),$$
$$\{F(x, t\psi, \psi, 1),\ -\infty < x < \infty\} \stackrel{\mathcal{D}}{=} \{F(x, t, 1, 1\},\ -\infty < x < \infty\}.$$

Theorem 3.1

$$\mathcal{M}_t(x, \psi, N) = \frac{1}{1 - (1 - p_{t/\psi})^N}\left[\left(1 - p_{t/\psi}(1 - \mathcal{M}_{t/\psi}(x))\right)^N - (1 - p_{t/\psi})^N\right] \tag{3.1}$$

$(t = \psi, 2\psi, \ldots)$.

Proof.

$$\begin{aligned}
&\mathcal{M}_t(x, \psi, N) \\
&= \mathbf{P}\{M(t, \psi, N) < xt^{1/2} \mid B^*(t, \psi, N) > 0\} \\
&= \sum_{k=1}^{N} \mathbf{P}\{M(t, \psi, N) < xt^{1/2}, B(t, \psi, N) = k \mid B(t, \psi, N) > 0\} \\
&= \frac{1}{\mathbf{P}\{B(t, \psi, N) > 0\}} \sum_{k=1}^{N} \mathbf{P}\{M(t, \psi, N) < xt^{1/2} \mid B(t, \psi, N) = k\}\mathbf{P}\{B(t, \psi, N) = k\} \\
&= \frac{1}{\mathbf{P}\{B(t, \psi, N) > 0\}} \sum_{k=1}^{N} (\mathcal{M}_t(x, \psi, 1))^k \mathbf{P}\{B(t, \psi, N) = k\} \\
&= \frac{1}{\mathbf{P}\{B(t, \psi, N) > 0\}} \sum_{k=1}^{N} (\mathcal{M}_{t/\psi}(x, 1, 1))^k \mathbf{P}\{B(t, \psi, N) = k\} \\
&= \frac{1}{1 - (1 - p_{t/\psi})^N} \sum_{k=1}^{N} (\mathcal{M}_{t/\psi}(x))^k \binom{N}{k} p_{t/\psi}^k (1 - p_{t/\psi})^{N-k},
\end{aligned}$$

which implies (3.1).

Theorem 3.2 *Let $t = t_N$ $(N = 1, 2, \ldots)$ be a sequence of positive numbers for which $Nt_N \to \infty$ as $N \to \infty$. Then we have*

$$\mathcal{M}_t(x, N^{-1}, N) \sim \frac{1}{e^{2/t} - 1} \left(\exp\left(\frac{2\mathcal{M}(x)}{t} \right) - 1 \right) \quad (N \to \infty), \tag{3.2}$$

where $\mathcal{M}(x)$ is defined in Theorem 2.1.

Proof. (3.2) follows from (3.1) and (2.4).

Remark 1.

$$\frac{1}{e^{2/t} - 1} \left(\exp\left(\frac{2\mathcal{M}(x)}{t} \right) - 1 \right) \longrightarrow \mathcal{M}(x) \quad (t \to \infty).$$

Remark 2. Let $x = x(t) > 0$ be a function of t for which $t^{-1}\mathcal{M}(x) \to \infty$ as $t \to 0$. Then

$$\frac{1}{c^{2/t} - 1} \left(\exp\left(\frac{2\mathcal{M}(x)}{t} \right) - 1 \right) \sim \exp\left(\frac{-2(1 - \mathcal{M}(x)))}{t} \right).$$

Hence

$$\mathcal{M}_t(x, N^{-1}, N) \sim \exp\left(\frac{-2(1 - \mathcal{M}(x))}{t} \right) \quad (t \to \infty),$$

provided that

$$Nt \to \infty \quad \text{and} \quad t^{-1}\mathcal{M}(x) \to \infty,$$

as $N \to \infty$ and $t \to 0$. For example, if

$$x = \left(2(1 \pm \varepsilon) \log \frac{1}{t}\right)^{1/2},$$

then by (2.5)

$$1 - \mathcal{M}(x) \sim C(d) \exp\left(-\frac{x^2}{2}\right) = C(d) t^{1\pm\varepsilon}.$$

Consequently,

$$\mathcal{M}_t((2(1 \pm \varepsilon) \log t)^{1/2}, N^{-1}, N) \sim \exp(-2C(d)t^{\pm\varepsilon}).$$

On combining Remarks 1 and 2, we conclude

$$\mathbf{P}\{M(t, N^{-1}, N) < xt^{1/2} \mid B^*(t, N^{-1}, N) > 0\} \to \mathcal{M}(x)$$

as $N \to \infty$ and $t \to \infty$, as well as

$$\mathbf{P}\{M(t, N^{-1}, N) < (2(1 \pm \varepsilon) t \log t^{-1})^{1/2} \mid B^*(t, N^{-1}, N) > 0\} \sim \exp(-2C(d)t^{\pm\varepsilon})$$

as $N \to \infty$, provided that $t \to 0$ and $Nt \to \infty$.

4. The case $\psi = 1$; a strong theorem for pre–super Brownian motions

Theorem 4.1 *Let t_N and x_N $(N = 1, 2, \ldots)$ be sequences of positive integers such that, as $N \to \infty$,*

$$t = t_N \uparrow \infty, \quad N^{-1} t_N \downarrow 0 \quad \text{and} \quad x = x_N \uparrow \infty.$$

Then

$$\mathcal{M}_t(x, 1, N) \sim \exp\left(-\frac{2N}{t}\left(1 - \mathcal{M}_t(x)\right)\right). \tag{4.1}$$

Proof. By Theorem 3.1 and (2.4) we have

$$\begin{aligned} &\mathcal{M}_t(x, 1, N) \\ &\sim \frac{1}{1 - (1 - \frac{2}{t})^N} \left[\left(1 - \frac{2}{t}\left(1 - \mathcal{M}_t(x)\right)\right)^N - \left(1 - \frac{2}{t}\right)^N\right]. \end{aligned}$$

Since, as $N \to \infty$,

$$\left(1 - \frac{2}{t}\right)^N \longrightarrow 0$$

and

$$\left(1-\frac{2}{t}\right)^N = o\left(\left(1-\frac{2}{t}\left(1-\mathcal{M}_t(x)\right)\right)^N\right),$$

we obtain

$$\begin{aligned}\mathcal{M}_t(x,1,N) &\sim \left(1-\frac{2}{t}\left(1-\mathcal{M}_t(x)\right)\right)^N \\ &\sim \exp\left(-\frac{2N}{t}\left(1-\mathcal{M}_t(x)\right)\right).\end{aligned}$$

Hence we have (4.1).

Corollary 4.1. *Let*

$$t = t_N = N^\alpha, \quad 0<\alpha<1, \quad \textit{and} \quad x = x_N = (2\beta\log N)^{1/2}, \ \beta>0.$$

Then, as $N\to\infty$, *we have*

$$\mathcal{M}_t(x,1,N) \sim \exp\left(-\,O(1)N^{1-\alpha-\beta}\right),$$

and, with $\varepsilon>0$,

$$\begin{aligned}&\mathbf{P}\Big\{M(t,1,N) \geq (2t(1-\alpha+\varepsilon)\log N)^{1/2}\Big\} \\ &\qquad\sim 1-\exp\left(-\,O(1)N^{-\varepsilon}\right) \sim O(1)N^{-\varepsilon}, \qquad (4.2)\\ &\mathbf{P}\Big\{M(t,1,N) < (2t(1-\alpha-\varepsilon)\log N)^{1/2}\Big\} \\ &\qquad\sim \exp\left(-\,O(1)N^{\varepsilon}\right). \qquad (4.3)\end{aligned}$$

Now we consider a sequence of pre–super Brownian motions with $\psi=1$ as follows. We let a pre–super Brownian motion starting with N initial particles from the origin $0\in\mathbb{R}^d$, run up to time N. Then we start an independent critical branching Wiener process from the same origin at the same time. Combining these two independent processes yields a pre–super Brownian motion with $N+1$ initial particles starting from the origin up to time $N+1$. In this context, the main conclusion of this section reads as follows.

Theorem 4.2. *We have*

$$\lim_{N\to\infty}\frac{M(N^\alpha,1,N)}{\left(2(1-\alpha)N^\alpha\log N\right)^{1/2}} = 1 \quad \textit{a.s.}$$

for any $0<\alpha<1$.

Proof. (4.3) clearly implies that

$$\liminf_{N\to\infty}\frac{M(N^\alpha,1,N)}{\left(2(1-\alpha)N^\alpha\log N\right)^{1/2}} \geq 1 \quad \text{a.s.} \qquad (4.4)$$

Let $N_k = k^K$, $K > \varepsilon^{-1}$. Then, by (4.2), we have

$$\limsup_{k\to\infty} \frac{M(N_k^\alpha, 1, N_k)}{(2(1-\alpha)N_k^\alpha \log N_k)^{1/2}} \le 1 \quad \text{a.s.} \tag{4.5}$$

Considering now $N_k \le N < N_{k+1}$, we arrive at

$$M(N^\alpha, 1, N) \le M(N^\alpha, 1, N_k) + M^*(N^\alpha, 1, N_{k+1} - N_k), \tag{4.6}$$

where M^* is obtained à la M from the $N_{k+1} - N_k$ independent critical branching Wiener processes that are added after time N_k to the pre–super Brownian motion starting with N_k initial particles. In view of (4.5) it is easy to see that we have

$$\limsup_{k\to\infty} \sup_{N_k \le N < N_{k+1}} \frac{M(N^\alpha, 1, N_k)}{(2(1-\alpha)N^\alpha \log N)^{1/2}} \le 1 \quad \text{a.s.}$$

and

$$\limsup_{k\to\infty} \sup_{N_k \le N < N_{k+1}} \frac{M(N^\alpha, 1, N_{k+1} - N_k)}{(2(1-\alpha)N^\alpha \log N)^{1/2}} = 0 \quad \text{a.s.}$$

Consequently, via (4.6) and (4.4) we arrive at Theorem 4.2.

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