CRITICAL BRANCHING WIENER PROCESS AND PRE–SUPER BROWNIAN MOTION

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Abstract. Here we study three kinds of branching models. In particular, the critical branching Wiener process and what we call the pre–super Brownian motion are conveniently dealt with via a simple branching Wiener process. Our main interest is to describe the asymptotic nature of distributions of the respective locations of particles that are produced by these processes.

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1 Introduction

In this paper we deal with three kinds of branching models of interest. One of these is the *critical branching Wiener process*, while another one is what we call the *pre–super Brownian motion* (cf. also [2]). We have found it convenient to study these two via a model that we call a *simple branching Wiener process*. Hence, we first describe this model, that will be our

Model I: *Simple branching Wiener process*

Let

$$\left\{U_{nk},\ k=1,2,\dots,2^{n-1},\ n=1,2,\dots\right\}$$

be an array of independent, uniform–[0,1] r.v.'s. Introduce the following notations:

$$\begin{aligned} V_{11} &= U_{11}, \\ V_{21} &= V_{11}U_{21}, \\ V_{22} &= V_{11}U_{22}, \end{aligned}$$

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$$
\begin{array}{rcl}
& \vdots & \\
V_{n,2k-1} & = & V_{n-1,k}U_{n,2k-1}, \\
V_{n,2k} & = & V_{n-1,k}U_{n,2k}, \quad (k=1,2,\ldots,2^{n-2},\ n=2,3,\ldots), \\
Y_{01} & = & 0, \\
Y_{nk} & = & 1-V_{nk}, \quad (k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots).
\end{array}
$$

$Y_{n,2k-1}$ and $Y_{n,2k}$ are called the daughters of $Y_{n-1,k}$ or, equivalently, $Y_{n-1,[(k+1)/2]}$ is the mother of $Y_{n,k}$. The array $\{Y_{nk},\ k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots\}$ is the family–tree of the offspring of Y_{01}.

Let

$$\left\{W_{nk}(\cdot),\ k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots\right\}$$

be an array of independent $I\!R^d$ valued Wiener processes on $[0,1]$, which is also independent from the array $\{U_{nk}\}$, and introduce the following notations:

$$
\begin{array}{rcl}
H_{11}(t) & = & W_{11}(t) \ \text{ if } 0\le t\le Y_{11}, \\
H_{21}(t) & = & \begin{cases} H_{11}(t) & \text{if } 0\le t\le Y_{11}, \\ H_{11}(Y_{11})+W_{21}(t-Y_{11}) & \text{if } Y_{11}\le t\le Y_{21}, \end{cases} \\
H_{22}(t) & = & \begin{cases} H_{11}(t) & \text{if } 0\le t\le Y_{11}, \\ H_{11}(Y_{11})+W_{22}(t-Y_{11}) & \text{if } Y_{11}\le t\le Y_{22}, \end{cases} \\
& \vdots & \\
H_{n,2k-1}(t) & = & \begin{cases} H_{n-1,k}(t) & \text{if } 0\le t\le Y_{n-1,k}, \\ H_{n-1,k}(Y_{n-1,k})+W_{n,2k-1}(t-Y_{n-1,k}) & \text{if } Y_{n-1,k}\le t\le Y_{n,2k-1}, \end{cases} \\
H_{n,2k}(t) & = & \begin{cases} H_{n-1,k}(t) & \text{if } 0\le t\le Y_{n-1,k}, \\ H_{n-1,k}(Y_{n-1,k})+W_{n,2k}(t-Y_{n-1,k}) & \text{if } Y_{n-1,k}\le t\le Y_{n,2k}, \end{cases}
\end{array}
$$

$(k=1,2,\ldots,2^{n-2},\ n=2,3,\ldots)$.

Let $\Lambda(t)$ $(0\le t<1)$ be the set of those (n,k) pairs of integers $(k=1,2,\ldots,2^{n-1}$, $n=1,2,\ldots)$ for which

$$Y_{n-1,[(k+1)/2]}\le t, \quad Y_{nk}>t,$$

and let

$$
\begin{array}{rcl}
Q(t) & = & \left\{H_{nk}(t): Y_{nk}>t,\ Y_{n-1,[(k+1)/2]}\le t\right\} = \{H_{nk}(t):(n,k)\in\Lambda(t)\}, \\
\mathcal{F}_t(R) & = & \#\{(n,k): H_{nk}(t)\in Q(t)\cap R\},
\end{array}
$$

where $0\le t<1$, and R is a Borel subset of $I\!R^d$. Furthermore, we introduce the family of empirical (random) measures

$$F_t(R)=(1-t)\mathcal{F}_t(R),$$

and with

$$R_x = \{y : y \in I\!R^d,\ y < x\},$$

we write

$$F_t(x) = F_t(R_x).$$

The latter is is the main object of our study in the first four sections.

Let $\mathcal{P}$ be the space of finite measures on $I\!R^d$ with the Lévy–Prokhorov distance $\rho(\cdot,\cdot)$.

Our main result in this model is Theorem 1.1, that will be proven in Section 3.

Theorem 1.1. *With any $u \in (1/2, 1)$ we have*

$$\begin{aligned} &\mathbf{P}\left\{\exists\ v \in [u,1)\ \textit{such that}\ \rho(F_u(\cdot), F_v(\cdot)) \geq 4(1-u)^{1/(d+3)}\right\} \\ &\leq\ 2^{d+2}\left(\log\frac{1}{1-u}\right)^d (1-u)^{1/(d+3)}. \end{aligned}$$

Consequently there exists a $\mathcal{P}$ valued random measure F such that

$$\begin{aligned} &\mathbf{P}\left\{\exists\ v \in [u,1)\ \textit{such that}\ \rho(F_v(\cdot), F(\cdot)) \geq 4(1-u)^{1/(d+3)}\right\} \\ &\leq\ 2^{d+2}\left(\log\frac{1}{1-u}\right)^d (1-u)^{1/(d+3)} \end{aligned}$$

and

$$\limsup_{u\uparrow 1}(1-u)^{-1/(d+3)}\rho(F_u, F) \leq 4 \quad a.s.$$

We note that the so far defined r.v.'s and processes define (live on) a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, which is also that of Theorem 1.1.

Let $A \subset \mathcal{P}$ be a Borel set and define the probability measures

$$\begin{aligned} \mu_u(A) &= \mathbf{P}\left\{F_u(\cdot) \in A\right\}, \quad 0 < u < 1, \\ \mu(A) &= \mathbf{P}\left\{F_u(\cdot) \in A\right\}. \end{aligned}$$

Let $\mathcal{M}$ be the set of probability measures defined on $\mathcal{P}$. Further let ρ be the Lévy–Prokhorov distance on $\mathcal{M}$.

Theorem 1.1 clearly implies

Theorem 1.2. *We have*

$$\rho(\mu_u, \mu) \leq 2^{d+3}\Big(\log\frac{1}{1-u}\Big)^d (1-u)^{1/(d+3)}$$

if $u > 1/2$.

Our next goal is to study the properties of the limit measure μ.

Let

$$B(r) = \left\{ F \in \mathcal{P} : \int_{T(r)} dF(x) \neq 0 \right\}$$

where

$$T(r) = I\!R^d \backslash [-r, r]^d.$$

In Section 4 we establish the following three theorems.

Theorem 1.3. *We have*

$$\mu(B(r)) \leq \exp\left(-\frac{1}{2}(r - 2r^{3/4})^2 \right)$$

if $r > 1$. *Furthermore,*

$$\mu\left\{ F : \int_{I\!R^d} dF(x) < t \right\} = 1 - e^{-t}, \quad t \geq 0,$$

and

$$\mathbf{E}F(x) = \int_{\mathcal{P}} F(x) d\mu = \Phi(x) = (2\pi)^{-d/2} \int_{R_x} \exp\Big(-\frac{u^2}{2} \Big) du, \quad x \in I\!R^d,$$

where $u^2 := \langle u, u \rangle$.

Theorem 1.4. *Let* F_1 *and* F_2 *be* $\mathcal{P}-$*valued i.i.d. random measures with distribution* μ, *and let*

$$\psi(x) = \frac{1}{2} \int_{I\!R^d} \int_0^1 \left(F^{(1)}\left(\frac{x-y}{(1-\alpha)^{1/2}} \right) + F^{(2)}\left(\frac{x-y}{(1-\alpha)^{1/2}} \right) \right) \phi_\alpha(y) d\alpha dy, \ x \in I\!R^d,$$

where

$$\phi_\alpha(y) = (2\pi\alpha)^{-d/2} \exp\Big(-\frac{y^2}{2\alpha} \Big), \ 0 < \alpha < 1, \ y \in I\!R^d.$$

Then ψ *is a* $\mathcal{P}-$*valued random measure with distribution* μ.

Theorems 1.3 and 1.4 describe two sets of properties of μ. Our next theorem claims that these two sets of properties determine μ uniquely.

Theorem 1.5. μ *is the only probability measure on* $\mathcal{P}$ *for which the statements of Theorems* 1.3 *and* 1.4 *hold true simultaneously.*

Now we are ready to introduce our second model that will be studied in Section 5.

Model II: *Critical Branching Wiener Process*

This model is featured as follows:

(i) a particle starts from the position $0 \in I\!R^d$ and executes a Wiener process $W(t) \in I\!R^d$,

(ii) arriving at time $t = 1$ to the new location $W(1)$, it dies,

(iii) at death it is replaced by Z offspring where

$$\mathbf{P}\{Z = 0\} = \mathbf{P}\{Z = 2\} = \frac{1}{2},$$

(iv) each offspring, starting from where its ancestor dies, executes a Wiener process (from its starting point) and repeats the above given steps, and so on. All Wiener processes and offspring–numbers are assumed to be independent of one–another.

Let

(a) $B(t)$ be the number of particles living at time t, the particles born at time t to be counted as alive at time t but not at time $t + 1$, i.e., $B(0) = 1$, $\mathbf{P}\{B(1) = 0\} = \mathbf{P}\{B(1) = 2\} = 1/2$,

(b) $X_{t1}, X_{t2}, \ldots, X_{t,B(t)}$ be the locations of the particles at time t,

(c) $$\lambda(A, t) = \#\{i: \ 1 \le i \le B(t), \ X_{ti} \in A\},$$

where A is a Borel set in $I\!R^d$ and $t = 0, 1, 2, \ldots$,

(d) $$\mathcal{G}(x, t) := \lambda(R_x, t),$$

(e) $$G(x, t) := t^{-1}\mathcal{G}(xt^{1/2}, t),$$

(f) μ_t be a probability measure on $\mathcal{P}$ defined by

$$\mu_t(A) = \mathbf{P}\{G(x, t) \in A | B(t) > 0\},$$

where $A \subset \mathcal{P}$ is a Borel set.

The sequence $\{B(t), \ t = 0, 1, 2, \ldots\}$ is called a (critical) branching process. The sequence $\lambda(A, t)$ of random measures is called a (critical) branching Wiener process.

Now we are ready to formulate our main result on this model. It will be proved in Section 5.

Theorem 5.1. *There exists a probability measure μ on $\mathcal{P}$ such that for any $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} \mathbf{P}\{\rho(\mu_t, \mu) \ge \varepsilon\} = 0,$$

and μ satisfies the statements of Theorems 1.3, 1.4 *and* 1.5.

Next follows the description of our third model that will be detailed and studied in Section 6.

Model III: *Pre-super Brownian motion*

(i) N ($N = 1, 2, \ldots$) particles start from the position $0 \in I\!R^d$ and execute N independent Brownian motions (Wiener processes) $W_1(t), W_2(t), \ldots, W_N(t)$ ($W_i(t) \in I\!R^d$, $0 \leq t < \infty$, $i = 1, 2, \ldots, N$),

(ii) arriving at time $t = \psi$ to the new locations $W_1(\psi), W_2(\psi), \ldots, W_N(\psi)$, they die,

(iii) at death they are replaced by $Z_1, Z_2, \ldots, Z_N$ offspring (respectively), where $Z_1, Z_2, \ldots, Z_N$ are i.i.d.r.v.'s (also independent from $W_i(t)$ ($i = 1, 2, \ldots, N$)) with

$$\mathbf{P}\{Z_i = 0\} = \mathbf{P}\{Z_i = 2\} = \frac{1}{2},$$

(iv) each offspring, starting from where its ancestor dies, executes a Brownian motion (Wiener process) (from its starting point, between $t = \psi$ and $t = 2\psi$) and repeats the above given steps. Wiener processes and offspring-numbers are assumed to be independent of one another.

Let

(a) $B(t, \psi, N)$ ($t = 0, \psi, 2\psi, \ldots$) be the number of particles living at time $t = i\psi$, the particles born at time $i\psi$ to be counted as being alive at time $i\psi$, but not at time $(i+1)\psi$, i.e., to begin with, for $i = 0, 1$, respectively, we have

$$B(0, \psi, N) = N,$$

$$\mathbf{P}\{B(\psi, \psi, N) = 2k\} = \binom{N}{k} 2^{-N} \quad (k = 0, 1, 2, \ldots, N),$$

(b) $B^*(t, \psi, N)$ ($t = 0, \psi, 2\psi, \ldots$) be the number of those particles (among the N ancestors) which have at least one living offspring at time t, i.e.,

$$B^*(0, \psi, N) = N,$$

$$\mathbf{P}\{B^*(\psi, \psi, N) = k\} = \binom{N}{k} 2^{-N} \quad (k = 0, 1, 2, \ldots, N).$$

Clearly, for any $t \geq \psi$, we have

$$0 \leq B(t, \psi, N) \leq N, \quad B(t, \psi, N) \geq 2B^*(t, \psi, N),$$
$$\{B(t, \psi, N) = 0\} = \{B^*(t, \psi, N) = 0\},$$

(c) $X_{t1}, X_{t2}, \ldots, X_{tB(t,\psi,N)}$ be the locations of the particles at time t in $I\!R^d$,

(d) $\lambda(A,t,\psi,N) := \#\{i : 1 \le i \le B(t,\psi,N),\ X_{ti} \in A\}$,

(e) $A(x) := \{y \in I\!R^d : y < x\},\ x \in I\!R^d$,

(f) $\lambda(x,t,\psi,N) := \lambda(A(x),t,\psi,N)$,

(g) $F(x,t,\psi,N) := N^{-1}\lambda(xt^{1/2},t,\psi,N)$.

On letting now $\psi = N^{-1}$, we are ready to formulate our main results on pre-super Brownian motion, which we will prove in Section 6.

Theorem 6.1. *For any $t > 0$ fixed, we have*

$$F(\cdot,t,N^{-1},N) \xrightarrow{\mathcal{L}} F^{(t)} \quad (N \to \infty)$$

on the set $\{B^*(t,N^{-1},N) > 0\}$.

Theorem 6.2. *Let $\{t_N,\ N = 1,2,\ldots\}$ be a sequence of positive numbers for which, as $N \to \infty$,*

$$t_N \to 0, \quad t_N N \to \infty.$$

Then we have

$$F(\cdot,t_N,N^{-1},N) \to \Phi(\cdot) \quad \textit{as } N \to \infty.$$

2 On the family tree $\{Y_{nk}\}$

At first we recall

Lemma 2.1. (Lemma 2.11 of [5]) *For any $0 < s < 1$ and $\ell = 1,2,\ldots$ we have*

$$\mathbf{P}\{L(s) = \ell\} = (1-s)s^{\ell-1},$$

where $L(s)$ is the cardinality of the set $\Lambda(s)$.

The next lemma is a trivial consequence of Lemma 2.1.

Lemma 2.2. *We have*

$$\begin{aligned}
\mathbf{E}L(s) &= (1-s)^{-1}, \\
\mathrm{Var}L(s) &= s(1-s)^{-2}, \\
\mathbf{E}(L(v)|L(u)) &= L(u)\,\frac{1-u}{1-v}, \\
\mathrm{Var}(L(v)|L(u)) &= L(u)\,\frac{(1-u)(v-u)}{(1-v)^2},
\end{aligned}$$

where $0 \le u < v < 1$.

Let

$$M(s) = (1-s)L(s), \quad (0 < s < 1).$$

Then Lemmas 2.1 and 2.2 easily imply the following conclusions.

Lemma 2.3. *$\{M(s),\ 0 \le s < 1\}$ is a martingale, i.e.,*

$$\mathbf{E}(M(v)|M(u)) = M(u) \quad (0 \le u < v < 1).$$

Furthermore, we have

$$\begin{aligned}
\mathbf{E}M(u) &= 1, \\
\mathrm{Var}M(u) &= u, \\
\mathbf{E}(M(u)M(v)|M(u)) &= (M(u))^2, \\
\mathbf{E}M(u)M(v) &= \mathbf{E}(M(u))^2 = \mathrm{Var}M(u) + 2 = 1 + u, \\
\mathbf{E}(M(v) - M(u))^2 &= v - u, \\
\mathbf{E}((M(v) - M(u))^2|M(u)) &= \mathrm{Var}(M(v)|M(u)) = (v-u)M(u).
\end{aligned}$$

Lemma 2.3 and well–known martingale theorems imply

Lemma 2.4. *The random variable*

$$M(1) := \lim_{u \uparrow 1} M(u)$$

exists almost surely, and

$$\begin{aligned}
\mathbf{E}M(1) &= 1, \\
\mathbf{E}(M(1))^2 &= 2, \\
\mathbf{E}(M(1) - M(u))^2 &= 1 - u, \\
\mathbf{E} \sup_{u \le v \le 1} (M(1) - M(v))^2 &\le 4(1-u), \\
\mathbf{E} \sup_{u \le v \le 1} (M(v) - M(u))^2 &\le 4(1-u), \\
\mathbf{E} \sup_{u \le v \le 1} (M(v) - M(u))^2|M(u)) &\le 4(1-u)M(u), \\
\mathbf{P}\left\{ \sup_{u \le v \le 1} |M(1) - M(v)| \ge 2\lambda(1-u)^{1/2} \right\} &\le \lambda^{-2}, \\
\mathbf{P}\left\{ \sup_{u \le v \le 1} |M(v) - M(u)| \ge 2\lambda(1-u)^{1/2} \right\} &\le \lambda^{-2}, \\
|M(1) - M(u)| &= O((1-u)^{1/2}), \ (u \uparrow 1), \quad a.s.
\end{aligned}$$

Lemma 2.1 and simple calculations imply

Lemma 2.5. *We have*

$$\mathbf{P}\{M(1) < x\} = \lim_{u\uparrow 1} \mathbf{P}\{M(u) < x\} = 1 - e^{-x}, \ x \geq 0.$$

Lemma 2.6. ([6], Lemma 2.10) *For any* $n = 1, 2, \ldots$ *and* $0 < \alpha < 1/8$ *we have*

$$\mathbf{P}\left\{\max_{1\leq k\leq 2^{n-1}} V_{nk} \geq (1-\alpha)^n\right\} \leq (8\alpha)^n.$$

Let

$$\begin{aligned} n(u) &= \min\left\{n : \max_{1\leq k\leq 2^{n-1}} V_{nk} < 1-u\right\} \\ &= 1 + \max\left\{n : \exists\ k \ni (n,k) \in \Lambda(u)\right\}, \ 0 \leq u < 1. \end{aligned}$$

Lemma 2.7. *Let* $C > 8$. *Then we have*

$$\mathbf{P}\left\{n(u) \geq C \log\frac{1}{1-u} + 2\right\} \leq \exp\left(-\left(\log\frac{C}{8}\right) C \log\frac{1}{1-u}\right)$$

for any $u \in [0,1)$.

Proof. Let

$$N = \left[C\log\frac{1}{1-u} + 2\right] - 1 > C\log\frac{1}{1-u},$$

and apply Lemma 2.6 with

$$\alpha = 1 - (1-u)^{1/N} < 1 - (1-u)^{\frac{1}{C\log(1-u)^{-1}}} = 1 - e^{-1/C} \leq \frac{1}{C}.$$

Then we obtain

$$\begin{aligned} \mathbf{P}\left\{\max_{1\leq k\leq 2^{N-1}} V_{Nk} \geq 1-u\right\} &= \mathbf{P}\left\{\max_{1\leq k\leq 2^{N-1}} V_{Nk} \geq (1-\alpha)^N\right\} \\ &\leq \left(\frac{8}{C}\right)^N \leq \exp\left(-\left(\log\frac{C}{8}\right) C \log\frac{1}{1-u}\right), \end{aligned}$$

which, in turn, implies Lemma 2.7.

3 Proofs of Theorems 1.1 and 1.2

We first note that

$$\begin{aligned}\mathcal{F}_t(I\!\!R^d) &= L(t),\\ F_t(I\!\!R^d) &= M(t),\\ \mathbf{E}F_t(R) &= \Phi(t,R) := (2\pi t)^{-d/2}\int_R \exp\Big(-\frac{x^2}{2t}\Big)dx,\ 0<t<1,\end{aligned}$$

where R is a Borel set of $I\!\!R^d$.

Introduce the following notations:

(i) let $\mathcal{N}(u,v,R)$ $(0<u<v<1,\ R\subset I\!\!R^d)$ be the number of those $H_{nk}(v)$'s whose ancestors (i.e., mothers, or grandmothers, or...) at time u are located in R,

(ii) let $\mathcal{A}(u,v,R_1,R_2)$ be the number of those $H_{nk}(u)$'s which are located in R_1 but who have at least one offspring located in R_2 at time v,

(iii)
$$C(x,r)=\{y: y\in I\!\!R^d,\ |x-y||\le r\},$$

(iv)
$$R^+(\varepsilon)=\bigcup_{x\in R} C(x,\varepsilon),$$

(v)
$$R^-(\varepsilon)=\bigcup_{\{x:C(x,\varepsilon)\subset R\}}\{x\}.$$

The next lemma is a simple conseqence of Lemmas 2.2 and 2.4.

Lemma 3.1. *We have*

$$\mathbf{E}(\mathcal{N}(u,v,R)|\mathcal{F}_u(R))=\frac{1-u}{1-v}\mathcal{F}_u(R),$$

$$\mathbf{E}(\sup_{u\le v<1}(\mathcal{N}(u,v,R)(1-v)-F_u(R))^2|\mathcal{F}_u(R))\le 4(1-u)F_u(R).$$

Applying Lemma 2.2 and some elementary properties of a Wiener process, we obtain

Lemma 3.2. *Let* $(n,k)\in\Lambda(v)$, $0<u<v<1$, $z>0$. *Then*

$$\mathbf{P}\Big\{\sup_{u\le t\le v}|H_{nk}(t)-H_{nk}(u)|\ge z(v-u)^{1/2}\Big\}\le \exp\Big(-\frac{z^2}{2}\Big),$$

and

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{(n,k)\in\Lambda(v)}\ \sup_{u\le t\le v}|H_{nk}(t)-H_{nk}(u)|\ge z(v-u)^{1/2}\Big\}\\
\le\ & \mathbf{EP}\Big\{\sup_{(n,k)\in\Lambda(v)}\ \sup_{u\le t\le v}|H_{nk}(t)-H_{nk}(u)|\ge z(v-u)^{1/2}|L(v)\Big\}\\
\le\ & \exp\Big(-\frac{z^2}{2}\Big)\mathbf{E}L(v)=(1-v)^{-1}\exp\Big(-\frac{z^2}{2}\Big).
\end{aligned}
$$

Lemma 3.3. *Let $x>1$. Then, for any $0<u<1$, we have*

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{u\le v<1}\ \sup_{(n,k)\in\Lambda(v)}|H_{nk}(v)-H_{nk}(u)|\ge(1-u)^{1/2}x\Big\}\\
=\ & \le(1-u)^{-1}\exp\Big(-\frac{1}{2}(x-2x^{3/4})^2\Big).
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
v_\ell &= 1-1(1-u)\alpha^\ell,\ (\ell=0,1,2,\ldots,\ 0<\alpha<1),\\
\alpha &= z^{-1/2},\\
z^\ell &= z+\ell.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\sum_{\ell=0}^{\infty}z_\ell(v_{\ell+1}-v_\ell)^{1/2} &= (1-u)^{1/2}(1-\alpha)^{1/2}z\sum_{\ell=0}^{\infty}\alpha^{\ell/2}+(1-u)^{1/2}(1-\alpha)^{1/2}\sum_{\ell=0}^{\infty}\ell\alpha^{\ell/2}\\
&= (1-u)^{1/2}\frac{(1-\alpha)^{1/2}}{1-\alpha^{1/2}}z+(1-u)^{1/2}\frac{(1-\alpha)^{1/2}\alpha^{1/2}}{(1-\alpha^{1/2})^2}\\
&\le (1-u)^{1/2}(z+2z^{3/4}),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\ell=0}^{\infty}(1-v_{\ell+1})^{-1}\exp\Big(-\frac{z_\ell^2}{2}\Big) &= (1-u)^{-1}\sum_{\ell=0}^{\infty}\alpha^{-(\ell+1)}\exp\Big(-\frac{z_\ell^2}{2}\Big)\\
&\le (1-u)^{-1}\exp\Big(-\frac{z^2}{2}\Big)\sum_{\ell=0}^{\infty}\alpha^{-(\ell+1)}\exp(-\ell z)\\
&= (1-u)^{-1}z^{1/2}\exp\Big(-\frac{z^2}{2}\Big)\sum_{\ell=0}^{\infty}(z^{1/2}e^{-z})^\ell\\
&\le 2z^{1/2}(1-u)^{-1}\exp\Big(-\frac{z^2}{2}\Big),
\end{aligned}
$$

and

$$\sup_{u\le v<1}\sup_{(n,k)\in\Lambda(v)}|H_{nk}(v)-H_{nk}(u)|\le\sum_{\ell=0}^{\infty}\sup_{v_\ell\le v<v_{\ell+1}}\sup_{(n,k)\in\Lambda(v_{\ell+1})}|H_{nk}(v)-H_{nk}(v_\ell)|.$$

Hence, by Lemma 3.2, we conclude

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{u\le v<1}\sup_{(n,k)\in\Lambda(v)}|H_{nk}(v)-H_{nk}(u)|\ge(1-u)^{1/2}(z+2z^{3/4})\right\}\\
&\le \mathbf{P}\left\{\sum_{\ell=0}^{\infty}\sup_{v_\ell\le v<v_{\ell+1}}\sup_{(n,k)\in\Lambda(v_{\ell+1})}|H_{nk}(v)-H_{nk}(v_\ell)|\ge\sum_{\ell=0}^{\infty}z_\ell(v_{\ell+1}-v_\ell)^{1/2}\right\}\\
&\le \sum_{\ell=0}^{\infty}\mathbf{P}\left\{\sup_{v_\ell\le v<v_{\ell+1}}\sup_{(n,k)\in\Lambda(v_{\ell+1})}|H_{nk}(v)-H_{nk}(v_\ell)|\ge z_\ell(v_{\ell+1}-v_\ell)^{1/2}\right\}\\
&\le \sum_{\ell=0}^{\infty}(1-v_{\ell+1})^{-1}\exp\left(-\frac{z_\ell^2}{2}\right)\le 2z^{1/2}(1-u)^{-1}\exp\left(-\frac{z^2}{2}\right),
\end{aligned}$$

which, in turn, yields also Lemma 3.3.

Let

$$m(v)=\max_{(n,k)\in\Lambda(v)}|H_{nk}(v)|.$$

Then, by Lemma 3.3, we conclude

Lemma 3.4.

$$\mathbf{P}\left\{\sup_{0\le v<1}m(v)\ge x\right\}\le\exp\left(-\frac{1}{2}(x-2x^{3/4})^2\right)$$

if $x>1$.

Lemma 3.5. *The limit*

$$m=\lim_{u\uparrow 1}m(u)$$

exists almost surely, and

$$\mathbf{P}\left\{|m(u)-m|\ge(1-u)^{1/2}x\right\}\le(1-u)^{-1}\exp\left(-\frac{1}{2}(x-2x^{3/4})^2\right),$$

and

$$\mathbf{P}\{m\ge x\}\le\exp\left(-\frac{1}{2}(x-2x^{3/4})^2\right)$$

if $x>1$ *and* $u>1/2$.

Now, we are to investigate the distance between the random distributions $F_u(R)$ and $F_v(R)$ ($0 < u < v < 1$, $R \subset I\!R^d$). The first step is immediate.

Lemma 3.6. *For any $\varepsilon > 0$ and $0 < u < v < 1$, we have*

$$\mathcal{N}(u,v,R^-(\varepsilon)) - \mathcal{A}(u,v,R^-(\varepsilon),\overline{R}) \le \mathcal{F}_v(R) \le \mathcal{N}(u,v,R^+(\varepsilon)) + \mathcal{A}(u,v,\overline{R^+(\varepsilon)},R),$$

where $\overline{R}$ is the complement of R in $I\!R^d$, i.e., $\overline{R} = I\!R^d \backslash R$.

The next lemma is an immediate consequence of Lemma 3.1.

Lemma 3.7. *We have*

$$\mathbf{E} \sup_{u \le v < 1} ((1-v)\mathcal{N}(u,v,R^-(\varepsilon)) - F_u(R^-(\varepsilon))^2 \le 4(1-u)\Phi(u,R^-(\varepsilon)),$$

and

$$\mathbf{E} \sup_{u \le v < 1} ((1-v)\mathcal{N}(u,v,R^+(\varepsilon)) - F_u(R^+(\varepsilon))^2 \le 4(1-u)\Phi(u,R^+(\varepsilon)).$$

Via Lemma 3.3 in turn, we conclude

Lemma 3.8. *We have*

$$\begin{aligned} &\mathbf{P}\left\{\sup_{u \le v < 1} \mathcal{A}(u,v,R^-(\varepsilon),\overline{R}) > 0\right\} \\ &\le \ (1-u)^{-1} \exp\left(-\frac{1}{2}\left(\frac{\varepsilon}{(1-u)^{1/2}} - \frac{2\varepsilon^{3/4}}{(1-u)^{3/8}}\right)^2\right) \end{aligned}$$

and

$$\begin{aligned} &\mathbf{P}\left\{\sup_{u \le v < 1} \mathcal{A}(u,v,\overline{R^-(\varepsilon)},R) > 0\right\} \\ &\le \ (1-u)^{-1} \exp\left(-\frac{1}{2}\left(\frac{\varepsilon}{(1-u)^{1/2}} - \frac{2\varepsilon^{3/4}}{(1-u)^{3/8}}\right)^2\right), \end{aligned}$$

provided that

$$(1-u)^{-1/2}\varepsilon > 1.$$

Let

$$\mathcal{B}(\lambda,u,v,\varepsilon,R) = \{F_u(R^-(\varepsilon)) - 2\lambda(1-u)^{1/2} \le F_v(R) \le F_u(R^+(\varepsilon)) + 2\lambda(1-u)^{1/2}\}.$$

Lemma 3.9. *Assume that*

$$(1-u)^{-1/2}\varepsilon > 1.$$

Then we have

$$\begin{aligned}
&\mathbf{P}\{\mathcal{B}(\lambda,u,v,\varepsilon,R),\ \forall v: u\le v<1\}\\
&\ge\ 1-\lambda^{-2}-(1-u)^{-1}\exp\left(-\frac{1}{2}\left(\frac{\varepsilon}{(1-u)^{1/2}}-\frac{2e^{3/4}}{(1-u)^{3/8}}\right)^2\right).
\end{aligned}$$

Proof. By Lemma 3.7 we have

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{u\le v<1}|(1-v)\mathcal{N}(u,v,R^-(\varepsilon))-F_u(R^-(\varepsilon))|\ge 2\lambda(1-u)^{1/2}\right\}\\
&\le\ \lambda^{-2}\Phi(u,R^-(\varepsilon))\le\lambda^{-2}.
\end{aligned}$$

By Lemmas 3.6 and 3.8

$$\begin{aligned}
&\mathbf{P}\left\{(1-v)\mathcal{N}(u,v,R^-(\varepsilon))\ne F_v(R)\right\}\\
&\le\ (1-u)^{-1}\exp\left(-\frac{1}{2}\left(\frac{\varepsilon}{(1-u)^{1/2}}-\frac{2e^{3/4}}{(1-u)^{3/8}}\right)^2\right)
\end{aligned}$$

which, in turn, implies Lemma 3.9.

Let

$$\begin{aligned}
\lambda &= (1-u)^{-\alpha}, \quad (0<\alpha<1/2),\\
\varepsilon &= (1-u)^{\beta}, \quad (0<\beta<1/2),
\end{aligned}$$

and, as before,

$$F_u(x)=F_u(R_x).$$

Via Lemma 3.9 we arrive at

Lemma 3.10. *For any* $x\in I\!\!R^d$ *we have*

$$\begin{aligned}
&\mathbf{P}\left\{\mathcal{B}(\lambda,u,v,\varepsilon,R_x)\ \forall v: u\le v<1\right\}\\
&\ge\ 1-(1-u)^{2\alpha}-(1-u)^{-1}\exp\left(-\frac{1}{2}((1-u)^{\beta-1/2}-2(1-u)^{3\beta/4-3/8})^2\right)\\
&\ge\ 1-2(1-u)^{2\alpha},
\end{aligned}$$

provided $u>1/2$.

Note that (cf. definition right above Lemma 3.9)

$$\begin{aligned}&\mathcal{B}(\lambda, u, v, \varepsilon, R_x)\\ &\quad = \Big\{F_u(x-\varepsilon) - 2(1-u)^{1/2-\alpha} \le F_v(x) \le F_u(x+\varepsilon) + 2(1-u)^{1/2-\alpha}\Big\}.\end{aligned}$$

We now wish to show that Lemma 3.10 holds true uniformly in x. In order to do so, introduce the following notations:

$$\begin{aligned} x(j_i) &= -\log\frac{1}{1-u} + j_i\varepsilon,\ \ (j_i = 0, 1, 2, \dots, \Big[\frac{2}{\varepsilon}\log\frac{1}{1-u}\Big],\ \ i = 1, 2, \dots, d),\\ x(\underline{j}) &:= x(j_1, j_2, \dots, j_d) = (x(j_1), x(j_2), \dots, x(j_d)) \in I\!R^d,\\ \mathcal{B}_{\underline{j}} &:= \Big\{\mathcal{B}(\lambda, u, v, \varepsilon, R_{x(\underline{j})}),\ \forall v : u \le v < 1\Big\}.\end{aligned}$$

Then, by Lemma 3.10, we have

Lemma 3.11. *Let $2\alpha > \beta d$ and $u > 1/2$. Then*

$$\mathbf{P}\Big\{\bigcap_{\underline{j}} \mathcal{B}_{\underline{j}}\Big\} \ge 1 - 2(1-u)^{2\alpha}\Big(\frac{2}{\varepsilon}\log\frac{1}{1-u}\Big)^d = 1 - 2^{d+1}\Big(\log\frac{1}{1-u}\Big)^d (1-u)^{2\alpha-\beta d}.$$

Let

$$\begin{aligned}\alpha &= \frac{1}{2} - \frac{1}{d+3},\\ \beta &= \frac{1}{d+3},\end{aligned}$$

and

$$x(\underline{j}) \le x < x(\underline{j+1}).$$

Observe that

$$\begin{aligned}&\Big\{F_u(x(\underline{j}) - \varepsilon) - 2(1-u)^{1/2-\alpha} \le F_v(x(\underline{j})) \le F_u(x(\underline{j}) + \varepsilon) + 2(1-u)^{1/2-\alpha}\ \forall \underline{j}\Big\}\\ &\quad \subset \Big\{F_u(x - 2\varepsilon) - 2(1-u)^{1/2-\alpha} \le F_v(x) \le F_u(x + 2\varepsilon) + 2(1-u)^{1/2-\alpha}\Big\}.\end{aligned}$$

Hence, by Lemma 3.11, we have

Lemma 3.12. *Let $2\alpha > \beta d$ and $u > 1/2$. Then*

$$\begin{aligned}
&\mathbf{P}\Big\{F_u(x-2\varepsilon) - 2(1-u)^{1/2-\alpha} \le F_v(x) \le F_u(x+2\varepsilon) + 2(1-u)^{1/2-\alpha}, \\
&\qquad \forall x : -\log\frac{1}{1-u} \le x \le \log\frac{1}{1-u},\ \forall v : u \le v < 1\Big\} \\
&\ge\ 1 - 2^{d+1}\left(\log\frac{1}{1-u}\right)^d (1-u)^{2\alpha-\beta d}.
\end{aligned}$$

Since, by Lemma 3.2, we have

$$\mathbf{P}\left\{\sup_{(n,k)\in\Lambda(u)} \sup_{0\le t\le u} |H_{nk}(t)| \ge \log\frac{1}{1-u}\right\} \le (1-u)^{-1}\exp\left(-\frac{1}{2u}\left(\log\frac{1}{1-u}\right)^2\right),$$

we conclude also the following statements.

Lemma 3.13. *Let $u > 1/2$ and $2\alpha > \beta d$. Then*

$$\begin{aligned}
&\mathbf{P}\{F_u(x-2\varepsilon) - 2(1-u)^{1/2-\alpha} \le F_v(x) \le F_u(x+2\varepsilon) + 2(1-u)^{1/2-\alpha}, \\
&\qquad \forall x \in \mathbb{R}^d,\ \forall v : u \le v < 1\} \\
&\ge\ 1 - 2^{d+1}\left(\log\frac{1}{1-u}\right)^d (1-u)^{2\alpha-\beta d} - (1-u)^{-1}\exp\left(-\frac{1}{2u}\left(\log\frac{1}{1-u}\right)^2\right) \\
&\ge\ 1 - 2^{d+2}\left(\log\frac{1}{1-u}\right)^d (1-u)^{2\alpha-\beta d}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbf{P}\Big\{F_u(x - 2(1-u)^{1/(d+3)}) - 2(1-u)^{1/(d+3)} \le F_v(x) \\
&\qquad \le F_u(x + 2(1-u)^{1/(d+3)}) + 2(1-u)^{1/(d+3)},\ \forall x \in \mathbb{R}^d,\ \forall v : u \le v < 1\Big\} \\
&\ge\ 1 - 2^{d+2}\left(\log\frac{1}{1-u}\right)^d (1-u)^{1/(d+3)}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\mathbf{P}\{\rho(F_u(\cdot), F_v(\cdot)) \ge 4(1-u)^{1/(d+3)}\ \forall v : u \le v < 1\} \\
&\le\ 2^{d+2}\left(\log\frac{1}{1-u}\right)^d (1-u)^{1/(d+3)}.
\end{aligned}$$

Lemma 3.13 clearly implies Theorems 1.1 and 1.2.

Lemma 3.14. *Let*

$$\tilde{\mathcal{F}}_u(x) = \#\{(n,k) : (n,k) \in \Lambda(u),\ H_{nk}(Y_{nk}) < x\},$$

$(1/2 < u < 1,\ x \in I\!R^d)$, *and put*

$$\tilde{F}_u(x) = (1-u)\tilde{\mathcal{F}}_u(x).$$

Then

$$\mathbf{P}\{\rho(F_u, \tilde{\mathcal{F}}_u) \geq (1-u)^{1/2} z\} \leq (1-u)^{-1} \exp\left(-\frac{1}{2}(z - 2z^{3/4})^2\right),$$

provided that $z > 1$.

Proof. By Lemma 3.3 we arrive at

$$\begin{aligned}&\mathbf{P}\{F_v(x-(1-u)^{1/2}z) \leq \tilde{F}_v(x) \leq F_v(x+(1-u)^{1/2}z)\ \forall v : u \leq v < 1\}\\ &\geq\ 1-(1-u)^{-1}\exp\left(-\frac{1}{2}(z-2z^{3/4})^2\right),\end{aligned}$$

which implies Lemma 3.14.

4 The properties of the limit measure μ

Proof of Theorem 1.3. It is an immediate consequence of Lemma 3.5.

Proof of Theorem 1.4. Clearly, there are two particles located at $H_{11}(Y_{11})$ at time Y_{11}. Consider the offspring of the first one at time $u > Y_{11}$. Let $\mathcal{F}_u^{(1)}(x)$ $(x \in I\!R^d)$ be the distribution of these particles, i.e.,

$$\begin{aligned}\mathcal{F}_u^{(1)}(x) &= \#\{(n,k) \in \Lambda(u) : H_{nk}(u) < x \text{ and } H_{nk}(u) \text{ is an}\\ &\quad\ \text{offspring of the first particle located at } H_{11}(Y_{11})\\ &\quad\ \text{at time } Y_{11}\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{F}_u^{(2)} &= \#\{(n,k) \in \Lambda(u) : H_{nk}(u) < x \text{ and } H_{nk}(u) \text{ is an}\\ &\quad\ \text{offspring of the second particle located at } H_{11}(Y_{11})\\ &\quad\ \text{at time } Y_{11}\}.\end{aligned}$$

Note that (given Y_{11} and $H_{11}(Y_{11})$)

$$F_u^{(1)}(x) = \frac{1-u}{1-Y_{11}}\mathcal{F}_u^{(1)}(x)$$

and

$$F_u^{(2)}(x) = \frac{1-u}{1-Y_{11}} \mathcal{F}_u^{(2)}(x)$$

are $\mathcal{P}$–valued i.i.d. random measures. Then

$$F_u(x) = \frac{1}{2}\left(F_u^{(1)}\left(\frac{x - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}\right) + F_u^{(2)}\left(\frac{x - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}\right)\right).$$

Let $F^{(1)}$ resp. $F^{(2)}$ be the limits of $F_u^{(1)}$ resp. $F_u^{(2)}$ as $u \uparrow 1$, i.e.,

$$\lim_{u\uparrow 1} \rho(F_u^{(1)}, F^{(1)}) = \lim_{u\uparrow 1} \rho(F_u^{(2)}, F^{(2)}) = 0 \quad \text{a.s.}$$

The existence of these limits follows from Theorem 1.1. Hence for any Borel set $A \subset \mathcal{P}$ we have

$$\begin{aligned} \mu(A) &= \mathbf{P}\{F \in A\} = \mathbf{EP}\{F \in A | Y_{11}, H_{11}(Y_{11})\} \\ &= \mathbf{P}\left\{\frac{1}{2}\int_{I\!R^d}\int_0^1 \left(F^{(1)}\left(\frac{x-y}{(1-\alpha)^{1/2}}\right) + F^{(2)}\left(\frac{x-y}{(1-\alpha)^{1/2}}\right)\right)\phi_\alpha(y) d\alpha dy \in A\right\} \end{aligned}$$

which, in turn, implies Theorem 1.4.

Proof of Theorem 1.5. Let

(i) ν be an arbitrary probability measure on $\mathcal{P}$ which satisfies the properties given by Theorems 1.3 and 1.4,

(ii) $Y_0^* < Y_1^* < \cdots$ be the ordered sample of the array $\{Y_{01}, Y_{nk},\ k = 1, 2, \ldots, 2^{n-1},\ n = 1, 2, \ldots\}$, i.e.,

$$\begin{aligned} Y_0^* &= Y_{01} = 0, \\ Y_1^* &= Y_{11}, \\ Y_2^* &= \min(Y_{21}, Y_{22}), \\ &\ \vdots \end{aligned}$$

(iii) $\{G_0, G_k^{(\ell)},\ k = 1, 2, \ldots,\ \ell = 1, 2\}$ be an array of $\mathcal{P}$–valued i.i.d. random measures with distribution ν.

Define a $\mathcal{P}$–valued stochastic process $\{\Gamma_u = \Gamma_u(x),\ 0 \le u < 1,\ x \in I\!R^d\}$ as follows. Let

$$\Gamma_u = G_0 \quad \text{if} \quad 0 \le u < Y_1^*,$$

$$\Gamma_u(x) = \frac{1}{2}\left(G_1^{(2)}\left(\frac{x - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}\right) + G_1^{(2)}\left(\frac{x - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}\right)\right)$$

if $Y_1^* \le u < Y_2^*$. Then, for any Borel set $A \subset \mathcal{P}$, we have

$$\begin{aligned}\nu\{\Gamma_u \in A\} &= \mathbf{E}\nu\{\Gamma_u \in A | Y_{11}, H_{11}(Y_{11})\} \\ &= \nu\left\{\frac{1}{2}\int_{\mathbb{R}^d}\int_0^1 \left(G_1^{(2)}\left(\frac{x-y}{(1-\alpha)^{1/2}}\right) + G_1^{(2)}\left(\frac{x-y}{(1-\alpha)^{1/2}}\right)\right)\phi_\alpha(y)d\alpha dy \in A\right\} \\ &= \nu(A),\end{aligned}$$

i.e., the distribution of Γ_u $(Y_1^* \le u < Y_2^*)$ is ν. Let $Y_2^* \le u < Y_3^*$ and, for the sake of simplicity, assume that $Y_2^* = Y_{21}$, say. Let

$$\begin{aligned}2\gamma_u &= G_2^{(1)}\left(\frac{x - H_{21}(Y_{21})}{(1-Y_{21})^{1/2}}\right) + G_2^{(2)}\left(\frac{x - H_{21}(Y_{21})}{(1-Y_{21})^{1/2}}\right) \\ &= G_2^{(1)}\left(\frac{\frac{x-H_{11}(Y_{11})}{(1-Y_{11})^{1/2}} - \frac{H_{21}(Y_{21})-H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}}{(1-Y_{21})^{1/2}(1-Y_{11})^{-1/2}}\right) \\ &\quad + G_2^{(2)}\left(\frac{\frac{x-H_{11}(Y_{11})}{(1-Y_{11})^{1/2}} - \frac{H_{21}(Y_{21})-H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}}{(1-Y_{21})^{1/2}(1-Y_{11})^{-1/2}}\right).\end{aligned}$$

Observe that the distribution of

$$(1-Y_{21})(1-Y_{11})^{-1} = V_{21}V_{11}^{-1} = U_{21}$$

is uniform–(0,1), and the distraibution of

$$\frac{H_{21}(Y_{21}) - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}$$

(given U_{21}) is

$$\mathcal{N}\left(0, U_{21}^{1/2}\right).$$

Hence the distribution of γ_u is equal to that of

$$G_1^{(1)}\left(\frac{x - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}\right).$$

Let

$$\Gamma_u = \frac{1}{2}\left(\gamma_u + G_1^{(2)}\left(\frac{x - H_{11}(Y_{11})}{(1-Y_{11})^{1/2}}\right)\right), \quad (Y_2^* \le u < Y_3^*).$$

Then the distribution of Γ_u is ν.

Continuing this procedure, we get the process Γ_u $(0 \le u < 1)$, and the distribution of Γ_u is ν for any $0 \le u < 1$.

Now we compare Γ_u and F_u. By Theorem 1.3 we have that

$$\lim_{u \uparrow 1} \rho(\Gamma_u, F_u) = 0 \quad \text{a.s.},$$

which implies that $\mu = \nu$. Hence we have Theorem 1.5.

5 Critical Branching Wiener Process: Proof of Theorem 5.1

This section is devoted to the proof of Theorem 5.1. Towards this end, at first we recall a few definitions and lemmas of [5] and [6].

For any $0 \le s < t$, let $Q(s,t)$ be the number of those particles which are living at time s and which have at least one offspring living at time t. Clearly

$$B(s) \ge Q(s,t), \qquad B(t) \ge Q(s,t),$$

$$\{Q(s,t) = 0\} = \{B(t) = 0\}, \quad (0 \le s \le t),$$

and $Q(s,t)$ is a nondecreasing function of s, $(0 \le s \le t)$, and $Q(0,t) = 1$, provided that $B(t) \ge 1$. Hence on the set $\{B(t) > 0\}$ we can define a r.v. $\nu = \nu_{11} = \nu_{11}(t)$ as follows:

$$\nu = \inf\{s : 0 < s \le t, \ \ Q(s,t) = 2\}.$$

At time ν we have two particles which have at least one offspring living at time t. The time ν will be called the first branching time of the process. The two particles born at time ν can be considered as the roots of two independent brnahcing processes living at least till time t (starting from ν). Let $\nu_{21} = \nu_{21}(t)$, resp. $\nu_{22} = \nu_{22}(t)$, be the first branching tiems of the branching processes starting from ν. Clearly $\nu < \nu_{2i} \le t$, $(i = 1,2)$. In case $\nu = t$, define $\nu_{2i} = t$. Note that in case $\nu = t - 1$ we have also $\nu_{2i} = t$.

We can say again that at times ν_{21} (resp. ν_{22}) we have two (resp. two) particles, and they can be considered as the roots of four independent branching processes living at least till time t. Let $\nu_{31} = \nu_{31}(t)$ (resp. $\nu_{32} = \nu_{32}(t)$) be the first branching times of the branching processes starting from ν_{21}. Similarly let $\nu_{33} = \nu_{33}(t)$ (resp. $\nu_{34} = \nu_{34}(t)$) be the first branching times of the branching processes starting from ν_{22}. Note that in case $\nu_{21} \ge t-1$ we have $\nu_{31} = \nu_{32} = t$ and in case $\nu_{22} \ge t - 1$ we have $\nu_{33} = \nu_{34} = t$.

In general, at time ν_{nk} $(k = 1, 2, \ldots, 2^{n-1})$, we have two particles and they can be considered as the roots of two independent branching processes living at least till time t (starting from ν_{nk}). Let $\nu_{n+1,2k-1} = \nu_{n+1,2k-1}(t)$, resp. $\nu_{n+1,2k} = \nu_{n+1,2k}(t)$, be the first branching times of the branching processes starting at ν_{nk}. Note that $\nu_{n+1,2k-1} = \nu_{n+1,2k} = t$ if $\nu_{nk} \ge t - 1$.

Now, we recall a few lemmas which describe the behaviour of the r.v.'s ν_{nk}.

Lemma 5.1. ([5], Lemma 7) *For any $k = 1, 2, \ldots, t-1$, we have*

$$\frac{t-k}{t} \le \mathbf{P}\{\nu_{11}(t) > k \mid B(t) > 0\} \le \frac{t-k}{t} + \frac{2\log t + 1}{t}.$$

Lemma 5.1 tells us that the r.v. $t^{-1}(\nu - t)$ is "essentially" uniformly distributed in $(0,1)$. The next lemma claims that $t^{-1}(\nu - t)$ can be approximated by a uniform–$(0,1)$ r.v. if the underlying probability space is rich enough. From now on we assume, without loss of generality, that this space is rich enough.

Lemma 5.2. ([5], Lemma 9) *There exists a sequence of uniform–*$(0,1)$ *r.v.'s* $\{U(t),\ t=1,2,\ldots\}$ *on the set* $\{B(t)>0\}$ *such that*

$$tU(t)-1\le \nu_{11}(t)\le tU(t)+2\log t+3.$$

Similar results can be obtained for any $\nu_{kt}(t)$. In fact we have

Lemma 5.3. ([5], Lemma 10) *For any* $t=1,2,\ldots$ *there exists an array*

$$\{U_{nk}(t),\ k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots\}$$

of independent uniform–$(0,1)$ *r.v.'s such that*

$$\begin{aligned} t(Y_{n+1,\ell}-Y_{nk})-n(2\log t+3)-1 &\le \nu_{n+1,\ell}-\nu_{nk}\\ \le\ \ t(Y_{n+1,\ell}-Y_{nk})+(2\log t+3)n,& \end{aligned}$$

and

$$tY_{nk}-n(2\log t+3)\le \nu_{nk}\le tY_{nk}+n,$$

where $\{Y_{nk}\}$ *is defined by* $\{U_{nk}\}$ *as in the* Introduction*, and* ℓ *is* $2k-1$ *or* $2k$*.*

For any $t=1,2,\ldots$, let

$$\{W_{nk}^{(t)}=W_{nk}(\cdot),\ k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots\}$$

be an array of independent $I\!R^d$–valued Wiener processes which is independent from both of the arrays

$$\{\nu_{nk}(t),\ k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots\}$$

and

$$\{U_{nk}(t),\ k=1,2,\ldots,2^{n-1},\ n=1,2,\ldots\}.$$

Introduce the following notations:

$$\begin{aligned}
J_{11}(s) &= W_{11}(s) && \text{if } 0\le s\le tY_{11},\\
J_{21}(s) &= \begin{cases} J_{11}(s) & \text{if } 0\le s\le tY_{11},\\ J_{11}(Y_{11})+W_{21}(s-tY_{11}) & \text{if } tY_{11}\le s\le tY_{21},\end{cases}\\
J_{22}(s) &= \begin{cases} J_{11}(s) & \text{if } 0\le s\le tY_{11},\\ J_{11}(Y_{11})+W_{22}(s-tY_{11}) & \text{if } tY_{11}\le s\le tY_{22},\end{cases}\\
J_{nk}(s) &= \begin{cases} J_{n-1,[(k+1)/2]}(s) & \text{if } 0\le s\le tY_{n-1,[(k+1)/2]},\\ J_{n-1,[(k+1)/2]}(s) & \\ \quad +W_{nk}(s-tY_{n-1,[(k+1)/2]}) & \text{if } tY_{n-1,[(k+1)/2]}\le s\le tY_{nk},\end{cases}
\end{aligned}$$

$$
\begin{aligned}
K_{11}(s) &= W_{11}(s) && \text{if } 0 \le s \le \nu_{11}, \\
K_{21}(s) &= \begin{cases} K_{11}(s) & \text{if } 0 \le s \le \nu_{11}, \\ K_{11}(Y_{11}) + W_{21}(s-\nu_{11}) & \text{if } \nu_{11} \le s \le \nu_{21}, \end{cases} \\
K_{22}(s) &= \begin{cases} K_{11}(s) & \text{if } 0 \le s \le \nu_{11}, \\ K_{11}(Y_{11}) + W_{22}(s-\nu_{11}) & \text{if } \nu_{11} \le s \le \nu_{22}, \end{cases}
\end{aligned}
$$

and so on.

Lemma 5.4. *Let*

$$Z_{nk} = W_{nk}(\nu_{nk} - \nu_{n-1,[(k+1)/2]}) - W_{nk}(t(Y_{nk} - Y_{n-1,[(k+1)/2]})).$$

Then, for any $x > 1$, we have

$$\mathbf{P}\{|Z_{nk}| \ge x(n(2\log t + 3))^{1/2}\} \le \exp\left(-\frac{x^2}{2}\right).$$

Consequently, for any $C > 0$, we have

$$\mathbf{P}\left\{\max_{n \le C\log t} \max_{1 \le k \le 2^{n-1}} |Z_{nk}| \ge C^{1/2}(\log t)^2\right\} \le \left(-\frac{(\log t)^2}{3}\right).$$

Now we wish to compare the processes J and K.

Let

$$k(1) = k,\ k(2) = [(k+1)/2], \ldots,\ k(i+1) = [(k(i)+1)/2],$$
$$y_j = t(Y_{n-j,k(j+1)} - Y_{n-j-1,k(j+2)}),$$

and

$$\nu_j = \nu_{n-j,k(j+1)} - \nu_{n-j-1,k(j+2)}.$$

Then we have

$$J_{nk}(tY_{nk}) = \sum_{j=0}^{n-2} W_{n-j,k(j+1)}(y_j) + W_{11}(tY_{11}),$$

and

$$K_{nk}(\nu_{nk}) = \sum_{j=0}^{n-2} W_{n-j,k(j+1)}(v_j) + W_{11}(\nu_{11}).$$

Hence

$$
\begin{aligned}
&|J_{nk}(tY_{nk}) - K_{nk}(\nu_{nk})| \\
&\quad \le \sum_{j=0}^{n-2} |W_{n-j,k(j+1)}(y_j) + W_{n-j,k(j+1)}(v_j)| + |W_{11}(tY_{11}) - W_{11}(\nu_{11})|,
\end{aligned}
$$

and by Lemma 5.4 we obtain

Lemma 5.5. *We have*

$$\mathbf{P}\left\{\max_{n\le C\log t}\ \max_{1\le k\le 2^{n-1}}|J_{nk}(tY_{nk})-K_{nk}(\nu_{nk})|\ge C^{3/2}(\log t)^3\right\}\le \exp\left(-\frac{(\log t)^2}{3}\right).$$

Let $\Lambda(u,t)$ $(0<u<t)$ be the set of those (n,k) pairs of integers for which

$$tY_{n-1,[(k+1)/2]}\le u,\quad tY_{nk}>u.$$

Similarly, let $\mathcal{M}(u,t)$ $(0<u<t)$ be the set of those (n,k) pairs of integers for which

$$\nu_{n-1,[(k+1)/2]}\le u,\quad \nu_{nk}>u.$$

Let

$$\begin{aligned} n(u,t) &= 1+\max\{n:\exists k\ni (n,k)\in\Lambda(u,t)\},\\ m(u,t) &= 1+\max\{n:\exists k\ni (n,k)\in\mathcal{M}(u,t)\}. \end{aligned}$$

Then, by Lemma 2.7, we conclude

Lemma 5.6. *Let* $C>8$. *Then we have*

$$\mathbf{P}\left\{n(u,t)\ge C\log\frac{1}{1-ut^{-1}}+2\right\}\le \exp\left(-\left(\log\frac{C}{8}\right)C\log\frac{1}{1-ut^{-1}}\right).$$

Let

$$\begin{aligned} \tilde{\mathcal{F}}_u(x,t) &= \#\{(n,k):(n,k)\in\Lambda(u,t),\ J_{nk}(tY_{nk})<xt^{1/2}\},\\ \mathcal{F}_u(x,t) &= \#\{(n,k):(n,k)\in\Lambda(u,t),\ J_{nk}(u)<xt^{1/2}\},\\ \tilde{F}_u(x,t) &= \left(1-\frac{u}{t}\right)\tilde{\mathcal{F}}_u(x,t),\\ F_u(x,t) &= \left(1-\frac{u}{t}\right)\mathcal{F}_u(x,t). \end{aligned}$$

Then, by Theorem 1.1 and Lemma 3.14, we arrive at

Lemma 5.7. *For any* $t>0$ *and* $u\in(1/2,1)$ *there exist a* $\mathcal{P}$*–valued random measure* $F(\cdot,t)$ *such that*

$$\begin{aligned} &\mathbf{P}\left\{\exists v\in[u,t]\ \textit{such that}\ \rho(F_v(\cdot,t),F(\cdot,t))\ge 4\left(1-\frac{u}{t}\right)^{1/(d+3)}\right\}\\ \le\ & 2^{d+2}\left(\log\frac{1}{1-ut^{-1}}\right)^{d}\left(1-\frac{u}{t}\right)^{1/(d+3)}, \end{aligned}$$

and

$$\mathbf{P}\left\{\exists v \in [u,t] \text{ such that } \rho(\tilde{F}_v(\cdot,t), F(\cdot,t)) \geq 4\left(1-\frac{u}{t}\right)^{1/(d+3)}\right\}$$
$$\leq \; 2^{d+2}\left(\log\frac{1}{1-ut^{-1}}\right)^d\left(1-\frac{u}{t}\right)^{1/(d+3)}.$$

Let $A \subset \mathcal{P}$ be a Borel set and define the measures

$$\begin{aligned}\mu_u(A,t) &= \mathbf{P}\{F_u(\cdot,t) \in A\},\\ \mu(A,t) &= \mathbf{P}\{F(\cdot,t) \in A\},\\ \tilde{\mu}_u(A,t) &= \mathbf{P}\{\tilde{F}_u(\cdot,t) \in A\}.\end{aligned}$$

Then Lemma 5.7 easily implies

Lemma 5.8. *For any $t > 0$ and $u > 1/2$, we have*

$$\rho(\mu_u(\cdot,t), \mu(\cdot,t)) \leq 2^{d+3}\left(\log\frac{1}{1-ut^{-1}}\right)^d\left(1-\frac{u}{t}\right)^{1/(d+3)},$$

and

$$\rho(\tilde{\mu}_u(\cdot,t), \mu(\cdot,t)) \leq 2^{d+3}\left(\log\frac{1}{1-ut^{-1}}\right)^d\left(1-\frac{u}{t}\right)^{1/(d+3)}.$$

It is also easy to see that the measure μ satisfies the statements of Theorems 1.3–1.5.
Let

$$\tilde{\mathcal{G}}_u(x,t) = \#\{(n,k) : (n,k) \in \Lambda(u,t), K_{nk}(\nu_{nk}) < xt^{1/2}\},$$

and

$$\tilde{G}_u(x,t) = \left(1-\frac{u}{t}\right)\tilde{\mathcal{G}}_u(x,t).$$

Lemma 5.9. *For any $\varepsilon > 0$ we have*

$$\lim_{u\uparrow t}\mathbf{P}\{\rho(\tilde{G}_u(\cdot,t), G(\cdot,t)) \geq \varepsilon\} = 0.$$

Proof. By Lemmas 5.3 and 5.6 we have

$$\mathbf{P}\left\{|\nu_{nk} - tY_{nk}| \leq \left(C\log\frac{1}{1-t^{-1}u}+2\right)(2\log t+3)\ \forall : (n,k) \in \Lambda(u,t)\right\}$$
$$\geq \; 1-\exp\left(-\left(\log\frac{C}{8}\right)C\log\frac{1}{1-t^{-1}u}\right).$$

Note also that

(i) $\{Y_{nk} - u,\ (n,k) \in \Lambda(u,t)\}$ are independent uniform–$(0, tu)$ r.v.'s,

(ii) if $B_{nk} > 0$ $((n,k) \in \Lambda(u,t))$ is the number of offspring (at time t) of the particle located at $K_{nk}(\nu_{nk})$ at time ν_{nk}, then B_{nk}'s are independent r.v.'s with

$$\mathbf{E}B_{nk} \sim \frac{t - \nu_{nk}}{2},$$

(iii) the probability that the distance between an offspring of the particle located at $K_{nk}(\nu_{nk})$, $((n,k) \in \Lambda(u,t))$, and its parent is more than $x(t-u)^{1/2}$, is less than $t\exp(-x^2/2)$.

The above statements clearly imply Lemma 5.9.

Lemma 5.10. *With any $\varepsilon > 0$, we have*

$$\lim_{u \uparrow t} \mathbf{P}\left\{\rho(\tilde{G}_u(\cdot,t), \tilde{F}_u(\cdot,t)) \geq \varepsilon\right\} = 0.$$

Proof. It is an immediate consequence of Lemmas 5.5 and 5.6.

Proof of Theorem 5.1. It follows from Lemmas 5.5, 5.6, 5.8 and 5.9 combined.

6 Pre–super Brownian motion: Proofs of Theorems 6.1 and 6.2

Inspired by [3] and [4], our model of pre-super Brownian motion was introduced in [2], where we studied a time sequence of exact distribution functions for the most-right vertex of the quadrant in $\mathbb{R}^d$ that is determined by the surviving particles, as well as an asymptotic form of these distributions when $\psi = N^{-1}$, and $N \to \infty$. There, in addition, we also established a strong theorem when $\psi = 1$ and $N \to \infty$.

For the sake of proving Theorems 6.1 and 6.2, we recall three lemmas.

Lemma 6.1. ([1]) *For any $t = 0, 1, 2, \ldots$, we have*

$$\begin{aligned} \mathbf{E}B(t) &= 1, \\ \mathbf{E}(B(t))^2 &= t + 1, \\ \lim_{t \to \infty} B(t) &= 0 \quad a.s. \end{aligned}$$

Lemma 6.2. ([6]) *For any $t = 0, 1, 2, \ldots$, we have*

$$\frac{2}{t + 2 + 2\log(t+1)} \leq p_t := \mathbf{P}\{B > 0\} \leq \frac{2}{t+2}.$$

Lemma 6.3. ([2])

$$\mathbf{P}\{B^*(t,\psi,N)=k\} = \begin{pmatrix} N \\ k \end{pmatrix} p_{t/\psi}^k (1-p_{t/\psi})^{N-k},$$

$(k = 0, 1, 2, \ldots, N,\ t = \psi, 2\psi, \ldots)$, *where* $p(\cdot)$ *is defined in* Lemma 6.2.

In the case $\psi = N^{-1}$, Lemma 6.3 implies

Lemma 6.4. *For any* t *fixed and* $k = 0, 1, 2, \ldots$, *as* $n \to \infty$, *we have*

$$\begin{aligned} \mathbf{P}\{B^*(t,N^{-1},N)=k\} &\sim \begin{pmatrix} N \\ k \end{pmatrix} \left(\frac{2}{tN}\right)^k \left(1-\frac{2}{tN}\right)^{N-k} \\ &\sim \frac{1}{k!}\left(\frac{2}{t}\right)^k \exp\left(-\frac{2}{t}\right). \end{aligned}$$

Let μ be the probability measure defined in Section 1, and let $F_1, F_2, \ldots$ be a sequence of $\mathcal{P}$–valued i.i.d. random measures with distribution μ. Further let π be a Poisson r.v. independent of $\{F_i\}$ with parameter $2(tN)^{-1}$. Consider the random measure

$$F^{(t)} = \frac{F_1 + F_2 + \cdots + F_\pi}{\pi}$$

on the set $\{\pi > 0\}$.

Now Theorems 6.1 and 6.2 follow by applying Theorem 5.1 and Lemma 6.4.

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