A Strong Markov Property For Set-Indexed Processes

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Abstract

We introduce adapted sets and optional sets and we study a type of strong Markov property for set-indexed processes, that can be associated with the sharp Markov property defined by Ivanoff and Merzbach (2000a).

Keywords: set-indexed processes; strong Markov property; sharp Markov property; adapted set; optional set.

1 Background and Preliminaries

In the classical theory, the notion of a Markov process is based on the representation of a process whose behaviour satisfies the hypothesis of "independence of the future from the past", or equivalently, "the absence of the after-effect" at any fixed moment of time; the process is said to be strong Markov if the property of the absence of the after-effect remains valid at any *random* moment of time. Markov processes indexed by discrete subsets of the real line (i.e. discrete totally ordered sets) always possess this property (Gihman and Skorohod, 1974, p. 86-88), while in the continuous case, one can give a criterion for the process to be strong Markov, and prove that in many important cases there exists a version of the process satisfying this criterion (Gihman and Skorohod, 1975, p. 60-62).

The literature is rather scarce when it comes to the strong Markov property for processes indexed by partially ordered sets, the main difficulties being: 1.

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to decide which is the right Markov property to work with; and 2. to find the appropriate analogue for the notion of stopping time, for the stopped σ-field, for the 'future events' σ-field and for the 'present' σ-field.

In what follows we will give a brief review of the subject in the literature, trying to use, as much as possible, a unified notation for various papers, which will be consistent with the notation that we subsequently use in this paper.

We will use the notation $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$, if the σ-fields $\mathcal{F}$ and $\mathcal{H}$ are conditionally independent given $\mathcal{G}$.

First we mention in passing that Wong and Zakai (1985) introduced briefly the notion of strong Markov property for path-parametrized processes; however, these are not examples of processes indexed by partially ordered sets.

In the case when the index set is a countable partially ordered set T, various authors seem to agree that the best way to define a **Markov chain** $X := (X_t)_{t\in T}$ is to say that for any $t \in T$, $\mathcal{A}_1(t) \perp \mathcal{A}_2(t) \mid \sigma(X_t)$, where $\mathcal{A}_1(t) := \sigma(\{X_{t'}; t' \leq t\})$ and $\mathcal{A}_2(t) := \sigma(\{X_{t'}; t' \geq t\})$. Cairoli and Dalang (1996) use the assumption that the underlying probabilistic model is a Markov chain to solve an optimal control problem; as our framework is different, we will not discuss this case here. On the other hand, Greenwood and Evstigneev (1990) introduced the notion of **splitting element** to denote a random element τ in T which has the property that for any $t \in T$ we can write $\{\tau = t\} = F_1 \cap F_2$ with $F_i \in \mathcal{A}_i(t); i = 1,2$; to any splitting element τ, they associated the σ-fields

$$\mathcal{A}_i(\tau) := \sigma(\{F \cap \{\tau = t\}; F \in \mathcal{A}_i(t), t \in T\});\ i = 1,2$$

The fact that is relevant for our discussion here is that any Markov chain X is strong Markov, in the sense that for any splitting element τ, $\mathcal{A}_1(\tau) \perp \mathcal{A}_2(\tau) \mid \sigma(\{\tau, X_\tau\})$.

The case of a very general uncountable partially ordered set does not seem to be discussed anywhere in the literature. Instead, three important particular situations are considered. These are in fact three increasing levels of generality; the present paper will address the most general of them.

The first level deals with the case when the index set is the Euclidean space $\mathbf{R}_+^2$ (or, more generally $\mathbf{R}_+^d$) and therefore it inherits the extra structure introduced by the total ordering of the coordinate axes. The **sharp Markov** property (with respect to a set $D \subseteq \mathbf{R}_+^2$) of a two-parameter process $X := (X_z)_{z\in\mathbf{R}_+^2}$ has been in the literature for a long time (first time introduced by Lévy, 1948) and it requires that $\mathcal{F}_D \perp \mathcal{F}_{D^c} \mid \mathcal{F}_{\partial D}$, where $\mathcal{F}_A := \sigma(\{X_z; z \in A\})$ for any $A \subseteq \mathbf{R}_+^2$. Merzbach and Nualart (1990) defined a **stopping line** as a random decreasing line L for which $\{z \leq L\} \in \mathcal{F}_z := \mathcal{F}_{[0,z]}\ \forall z \in \mathbf{R}_+^2$, where a 'decreasing line' l is in fact the boundary of a lower set $D(l) \subseteq \mathbf{R}_+^2$; the stopped σ-field associated to any stopping line L is

$$\mathcal{F}_{D(L)} := \{F \in \mathcal{F}; F \cap \{L \leq l\} \in \mathcal{F}_{D(l)} \text{ for any decreasing line } l\}$$

For each random set α, the same authors defined the σ-field

$$\mathcal{F}_{\alpha}^{X} := \sigma(\{X_z 1_{\alpha}(z), 1_{\alpha}(z); z \in \mathbf{R}_{+}^{2}\})$$

and they showed that in fact $\mathcal{F}_{D(L)}^{X} = \mathcal{F}_{D(L)}$. The important fact proved by Merzbach and Nualart (1990) is that certain point processes, which are sharp Markov with respect to the sets $D(l)$, have the property that for any stopping line L, $\mathcal{F}_{D(L)} \perp \mathcal{F}_{D(L)^c}^{X} \mid \mathcal{F}_{L}^{X}$.

The second level of generality deals with the case of processes indexed by a collection of closed subsets of a d-dimensional Euclidean space. When this collection contains all closed subsets of the space, Evstigneev (1977) introduced a type of Markov property for general 'random fields' i.e., for filtrations $(\mathcal{F}_A)_A$ (which can arise in particular from a process $X := (X_A)_A$, if we set $\mathcal{F}_B := \sigma(\{X_A; A \subseteq B, A \text{ closed}\})$). More precisely, a random field $(\mathcal{F}_A)_A$ (or, in particular a process X) is called **locally Markov**, if for any disjoint closed sets A and B, $\mathcal{F}_A \perp \mathcal{F}_B \mid \mathcal{F}_{\partial A}$, or equivalently if for any closed set A, $\mathcal{F}_A \perp \mathcal{F}_{A^c} \mid \mathcal{F}_{\partial A}$. A closed-valued random set α is called a **Markov random set** if $\{\alpha \subseteq A\} \in \mathcal{F}_A$ for any closed set A; to any Markov random set α one can associate the σ-fields $\mathcal{A}_{\alpha} := \cap_{\epsilon>0}\mathcal{A}_{\alpha}^{\epsilon}$ and $\mathcal{B}_{\alpha} = \cap_{\epsilon>0}\mathcal{B}_{\alpha}^{\epsilon}$, where

$$\mathcal{A}_{\alpha}^{\epsilon} = \sigma(\{F \cap \{\alpha_{\epsilon} \supseteq A\}; F \in \mathcal{F}_A, A \text{ closed}\}) \vee \sigma(\alpha)$$

$$\mathcal{B}_{\alpha}^{\epsilon} = \sigma(\{F \cap \{(\partial\alpha)_{\epsilon} \supseteq A\}; F \in \mathcal{F}_A, A \text{ closed}\}) \vee \sigma(\alpha)$$

The main result of Evstigneev (1977) is that any locally Markov random field (or process) has the property that for any compact-valued Markov set α, for any closed set A and for any $F \in \mathcal{F}_A$, we have $P[F|\mathcal{A}_{\alpha}] = P[F|\mathcal{B}_{\alpha}]$ a.s. on the set $\{\omega; \alpha(\omega) \cap A = \emptyset\}$.

The third level of generality deals with the case of processes indexed by a collection $\mathcal{A}$ of compact subsets of a Hausdorff topological space T, collection which does not contain disjoint (non-empty) sets, is a semilattice (i.e. closed under arbitrary intersections), and *separable from above*, in a sense which will be specified below. The general theory of these processes was initiated and developed by Ivanoff and Merzbach in the late 80's and it produced impressive results in the martingale case (e.g. Ivanoff and Merzbach, 2000b). In the present paper we will consider a type of strong Markov property for these set-indexed processes, which can be associated to the 'sharp Markov property', one of the various types of Markov properties that have been introduced in this framework (e.g. Balan and Ivanoff, to appear; Ivanoff and Merzbach, 2000a).

The **separability from above** property of the indexing collection $\mathcal{A}$ allows us to approximate from above a set $A \in \mathcal{A}$ as

$$A = \cap_n g_n(A); \quad g_{n+1}(A) \subseteq g_n(A), A \subseteq g_n(A)^0 \ \forall n$$

where the approximation set $g_n(A)$ can be written as a finite union of sets that lie in a finite sub-semilattice $\mathcal{A}_n$ of $\mathcal{A}$; moreover, $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ $\forall n$ and g_n preserves arbitrary intersections and finite unions i.e. $g_n(\cap_{\alpha\in\Lambda}A_\alpha) = \cap_{\alpha\in\Lambda}g_n(A_\alpha),\ \forall A_\alpha \in \mathcal{A}$; and $\cup_{i=1}^k A_i = \cup_{j=1}^m A'_j \Rightarrow \cup_{i=1}^k g_n(A_i) = \cup_{j=1}^m g_n(A'_j),\ \forall A_i, A'_j \in \mathcal{A}$; note that this implies that the function g_n is monotone i.e. $A, A' \in \mathcal{A}, A \subseteq A' \Rightarrow g_n(A) \subseteq g_n(A')$. By convention, $g_n(\emptyset) = \emptyset$.

There are many examples of classes of sets which have these properties, from which one can recognize easily the case of processes indexed by the lower sets of the d-dimensional space $\mathbf{R}_+^d$ (A is a 'lower set' if $z \in A$ implies $[0, z] \subseteq A$), and in particular the case of processes indexed by the 'rectangles' $[0, z], z \in \mathbf{R}_+^d$ (or equivalently, the case of multi-parameter processes).

In this framework $\mathcal{A}(u)$ denotes the class of all finite unions of sets in $\mathcal{A}$, $\mathcal{C}$ is the semi-algebra of all sets of the form $C = A \backslash B, A \in \mathcal{A}, B \in \mathcal{A}(u)$, and $\mathcal{C}(u)$ is the algebra of all finite unions of sets in $\mathcal{C}$. Note that the function g_n can be extended to $\mathcal{A}(u)$ by setting $g_n(B) := \cup_{A\in\mathcal{A},A\subseteq B}g_n(A), B \in \mathcal{A}(u)$; the extension preserves finite unions and finite intersections and is monotone.

All the processes $X := (X_A)_{A\in\mathcal{A}}$ are assumed to have a unique additive extension to $\mathcal{A}(u), \mathcal{C}$ and $\mathcal{C}(u)$ i.e., whenever the set $B \in \mathcal{A}(u)$ can be written as $B = \cup_{i=1}^n A_i = \cup_{j=1}^m A'_j$ with $A_1, \ldots, A_n, A'_1, \ldots, A_m \in \mathcal{A}$

$$\sum_{i=1}^n X_{A_i} - \sum_{1\le i_1<i_2\le n} X_{A_{i_1}\cap A_{i_2}} + \cdots + (-1)^{n+1} X_{A_1\cap\ldots\cap A_n} =$$
$$\sum_{j=1}^m X_{A'_j} - \sum_{1\le j_1<j_2\le m} X_{A'_{j_1}\cap A'_{j_2}} + \cdots + (-1)^{m+1} X_{A'_1\cap\ldots\cap A'_m} \text{ a.s.;}$$

whenever the set $C \in \mathcal{C}$ can be written as $C = A\backslash B = A'\backslash B'$ with $A, A' \in \mathcal{A}; B, B' \in \mathcal{A}(u)$

$$X_A - X_{A\cap B} = X_{A'} - X_{A'\cap B'} \text{ a.s.;}$$

and the additive extension to $\mathcal{C}(u)$ is defined in the obvious manner.

A process $X := (X_A)_{A\in\mathcal{A}}$ is said to be monotone outer-continuous if for any decreasing sequence $(A_n)_n \subseteq \mathcal{A}$, $X_{\cap_n A_n} = \lim_n X_{A_n}$. Note that by additivity, X is monotone outer-continuous as well for a decreasing sequence $(B_n)_n \subseteq \mathcal{A}(u)$, provided that $\cap_n B_n \in \mathcal{A}(u)$.

An increasing collection $(\mathcal{F}_A)_{A\in\mathcal{A}}$ of σ-fields is called a filtration; an $\mathcal{A}$-indexed filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ can be extended to a filtration indexed by $\mathcal{A}(u)$ by defining $\mathcal{F}_B := \bigvee_{A\in\mathcal{A},A\subseteq B}\mathcal{F}_A, B \in \mathcal{A}(u)$. A filtration $(\mathcal{F}_B)_{B\in\mathcal{A}(u)}$ is called monotone outer-continuous if for any decreasing sequence $(B_n)_n \subseteq \mathcal{A}(u)$ with $\cap_n B_n \in \mathcal{A}(u)$, we have $\mathcal{F}_{\cap_n B_n} = \cap_n \mathcal{F}_{B_n}$.

A process $X := (X_A)_{A\in\mathcal{A}}$ is adapted with respect to a filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ if X_A is $\mathcal{F}_A$-measurable $\forall A \in \mathcal{A}$ (by additivity this implies that X_B is $\mathcal{F}_B$-measurable $\forall B \in \mathcal{A}(u)$). The minimal filtration with respect to which a process X is adapted is given by $\mathcal{F}_B := \sigma(\{X_A; A \in \mathcal{A}, A \subseteq B\}), B \in \mathcal{A}(u)$.

According to Ivanoff and Merzbach (2000a), a process $X := (X_A)_{A\in\mathcal{A}}$ is called **sharp Markov** if $\mathcal{F}_B \perp \mathcal{F}_{B^c} \mid \mathcal{F}_{\partial B}$, where $\mathcal{F}_{\partial B} := \sigma(\{X_A; A \in \mathcal{A}, A \subseteq B, A \not\subseteq B^0\}), \mathcal{F}_{B^c} := \sigma(\{X_A; A \in \mathcal{A}, A \not\subseteq B\})$.

The notion of **stopping set** was introduced by Ivanoff and Merzbach (1995) to denote an $\mathcal{A}(u)$-valued random set ξ which can be written as the finite union of some $\mathcal{A}$-valued random sets and has the property that $\{\xi \supseteq A\} \in \mathcal{F}_A, \forall A \in \mathcal{A}$ and $\{\xi = \emptyset\} \in \mathcal{F}_{\emptyset}$.

Unfortunately, the approximation $g_n(\xi)$ of a stopping set ξ may not be a stopping set, and this is the point where Ivanoff and Merzbach (2000a) were forced to introduce the additional assumptions of the monotone outer-continuity of the process and of the filtration. These assumptions are satisfied by point processes but they are not satisfied by many other important classes of processes, like the Brownian motion, for instance.

In the present paper we will be able to get around this difficulty by replacing the stopping sets with random sets called 'adapted sets', respectively 'optional sets', which are defined using the set-inclusion in the natural direction, not in the reverse direction as with the stopping sets. Using optional sets, we will obtain the same results as Ivanoff and Merzbach (2000a), which will be applied this time to a much broader class of processes. The importance of this work is that it indicates that the natural generalization of a Markov time is an optional set, and not a stopping set.

2 The Strong Markov Property

In this section we will introduce the adapted sets and the optional sets and we will study a type of strong (sharp) Markov property that can be associated to these objects.

We will assume that the approximating functions g_n have the following definition:

$$g_n(B) := \cap_{D \in \mathcal{A}_n(u); B \subseteq D^0} D, \quad \forall B \in \mathcal{A}(u)$$

This is the case of many examples of indexing collections, including the lower layers of $\mathbf{R}_+^d$ and the rectangles $[0, z], z \in \mathbf{R}_+^d$.

¿From the separability from above of the indexing collection $\mathcal{A}$ we have

$$\forall D, B \in \mathcal{A}(u), D \subseteq B^0 \ \Rightarrow \ \exists n \text{ such that } g_n(D) \subseteq B^0 \tag{1}$$

Let $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ be the extension to $\mathcal{A}(u)$ of a set-indexed filtration and $(\mathcal{F}_B^r)_{B \in \mathcal{A}(u)}$ its minimal outer-continuous filtration, defined by $\mathcal{F}_D^r := \cap_n \mathcal{F}_{g_n(D)}$, $D \in \mathcal{A}(u)$. Then

$$\mathcal{F}_D^r = \cap_{B \in \mathcal{A}(u), D \subseteq B^0} \mathcal{F}_B, \quad D \in \mathcal{A}(u) \tag{2}$$

The following assumption gives the approximation from below for a set in $\mathcal{A}(u)$.

Assumption 2.1 *For any $B \in \mathcal{A}(u)$ there exists a monotone increasing sequence $(d_n(B))_{n \geq 1} \subseteq \mathcal{A}(u)$ such that $B^0 = \cup_n d_n(B), d_n(B) \subseteq d_{n+1}(B)^0 \ \forall n$.*

Consequently,

$$\forall D, B \in \mathcal{A}(u), D \subseteq B^0 \;\Rightarrow\; \exists n \text{ such that } D \subseteq d_n(B)^0 \tag{3}$$

A *random set* α is a function with values in $\mathcal{A}(u)$, defined on a measurable space $(\Omega, \mathcal{F})$.

Definition 2.2 *A random set α is called an* **adapted set** *of the filtration $(\mathcal{F}_B)_{B\in\mathcal{A}(u)}$ if $\{\alpha \subseteq B\} \in \mathcal{F}_B$ for every $B \in \mathcal{A}(u)$; it is called an* **optional set** *of the filtration $(\mathcal{F}_B)_{B\in\mathcal{A}(u)}$ if $\{\alpha \subseteq B^0\} \in \mathcal{F}_B$ for every $B \in \mathcal{A}(u)$.*

To any adapted set α we can associate the σ-field:

$$\mathcal{F}_\alpha \stackrel{\text{def}}{=} \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B\} \in \mathcal{F}_B \; \forall B \in \mathcal{A}(u)\}$$

To any optional set α we can associate the σ-field:

$$\mathcal{F}^r_\alpha \stackrel{\text{def}}{=} \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B^0\} \in \mathcal{F}_B \; \forall B \in \mathcal{A}(u)\}$$

The following facts are completely analogous to the classical case.

Lemma 2.3 (a) *If $\alpha \equiv B \in \mathcal{A}(u)$, then α is an adapted set and $\mathcal{F}_\alpha = \mathcal{F}_B, \mathcal{F}^r_\alpha = \mathcal{F}^r_B$.*

(b) *A random set is an optional set of the filtration $(\mathcal{F}_B)_{B\in\mathcal{A}(u)}$ if and only if it is an adapted set of the filtration $(\mathcal{F}^r_B)_{B\in\mathcal{A}(u)}$. In particular, any adapted set is an optional set.*

(c) *If α is an optional set, then $\mathcal{F}^r_\alpha = \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B\} \in \mathcal{F}^r_B \; \forall B \in \mathcal{A}(u)\}$. In particular, if α is an adapted set then $\mathcal{F}_\alpha \subseteq \mathcal{F}^r_\alpha$.*

(d) *If α is an optional set and β is an adapted set such that $\alpha \subseteq \beta^0$, then $\mathcal{F}^r_\alpha \subseteq \mathcal{F}_\beta$.*

Proof: **(a)** Clear.

(b) If α is an optional set of the filtration $(\mathcal{F}_B)_{B\in\mathcal{A}(u)}$, then $\{\alpha \subseteq B\} = \cap_{n\geq m}\{\alpha \subseteq g_n(B)^0\} \in \mathcal{F}_{g_m(B)} \forall m \geq 1$ and hence $\{\alpha \subseteq B\} \in \cap_m \mathcal{F}_{g_m(B)} = \mathcal{F}^r_B$. Conversely, if α is an adapted set of the filtration $(\mathcal{F}^r_B)_{B\in\mathcal{A}(u)}$, then $\{\alpha \subseteq B^0\} = \cup_n\{\alpha \subseteq d_n(B)\} \in \mathcal{F}_B$, because $\{\alpha \subseteq d_n(B)\} \in \mathcal{F}^r_{d_n(B)} \subseteq \mathcal{F}_B$.

(c) Same type as argument as (b).

(d) For each $F \in \mathcal{F}^r_\alpha$ we have $F \cap \{\beta \subseteq B\} = (F \cap \{\alpha \subseteq B^0\}) \cap \{\beta \subseteq B\} \in \mathcal{F}_B \; \forall B \in \mathcal{A}(u)$ i.e. $F \in \mathcal{F}_\beta$. $\square$

Comment 2.4 If α is a discrete random set i.e., it takes on only countably many configurations, then α is an adapted set if and only if $\{\alpha = B\} \in \mathcal{F}_B \; \forall B \in \mathcal{A}(u)$. In this case $\mathcal{F}_\alpha = \{F \in \mathcal{F} : F \cap \{\alpha = B\} \in \mathcal{F}_B \; \forall B \in \mathcal{A}(u)\}$.

Proposition 2.5 *If α is an optional set then $g_n(\alpha)$ is a (discrete) adapted set and $\mathcal{F}^r_\alpha = \cap_n \mathcal{F}_{g_n(\alpha)}$.*

Proof: We claim that the following relation holds:

$$\{g_n(\alpha) = B\} = \{\alpha \subseteq B^0\} \setminus \bigcup_{D \in \mathcal{A}_n(u), D \subset B} \{\alpha \subseteq D^0\} \quad \forall B \in \mathcal{A}(u) \tag{4}$$

where $\subset$ denotes the strict inclusion.

(Suppose that $g_n(\alpha) = B$. Clearly $\alpha \subseteq g_n(\alpha)^0 = B^0$. If there were some $D \in \mathcal{A}_n(u), D \subset B$ such that $\alpha \subseteq D^0$, then by the definition of g_n we would have $g_n(\alpha) \subseteq D$, which is impossible. Conversely, suppose that $\alpha \subseteq B^0$ and $\alpha \not\subseteq D^0$ $\forall D \in \mathcal{A}(u), D \subset B$. Since $\alpha \subseteq B^0$, there exists an n such that $g_n(\alpha) \subseteq B$. Since $g_n(\alpha) \in \mathcal{A}(u)$ and $\alpha \subseteq g_n(\alpha)^0$, we cannot have $g_n(\alpha) \subset B$; hence $g_n(\alpha) = B$.)

¿From (4) it follows that $\{g_n(\alpha) = B\} \in \mathcal{F}_B$ $\forall B \in \mathcal{A}(u)$ i.e., $g_n(\alpha)$ is an adapted set.

Let us prove now that $\mathcal{F}^r_\alpha = \cap_n \mathcal{F}_{g_n(\alpha)}$. Since $\alpha \subseteq g_n(\alpha)^0$, we have $\mathcal{F}^r_\alpha \subseteq \mathcal{F}_{g_n(\alpha)}$. Conversely, let $F \in \cap_n \mathcal{F}_{g_n(\alpha)}$. For each $B \in \mathcal{A}(u)$ $F \cap \{\alpha \subseteq B\} = \cap_{n \geq m}(F \cap \{g_n(\alpha) \subseteq g_n(B)\}) \in \mathcal{F}_{g_m(B)} \forall m \geq 1$; hence $F \cap \{\alpha \subseteq B\} \in \cap_m \mathcal{F}_{g_m(B)} = \mathcal{F}^r_B$ i.e. $F \in \mathcal{F}^r_\alpha$. □

The following result is completely analogous to processes indexed by $\mathbf{R}_+$.

Proposition 2.6 *If $X := (X_A)_{A \in \mathcal{A}}$ is a monotone outer-continuous process, which is adapted with respect to the filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$, then*

(a) *X_α is $\mathcal{F}_\alpha$-measurable for any discrete adapted set α; and*

(b) *X_α is $\mathcal{F}^r_\alpha$-measurable for any optional set α.*

Proof: **(a)** For any arbitrary $a \in \mathbf{R}$, $\{X_\alpha < a\} \cap \{\alpha = B\} = \{X_B < a\} \cap \{\alpha = B\} \in \mathcal{F}_B, \forall B \in \mathcal{A}(u)$ i.e., $\{X_\alpha < a\} \in \mathcal{F}_\alpha$.

(b) By the monotone outer-continuity of the process, $X_\alpha = \lim_n X_{g_n(\alpha)}$ and $\{X_\alpha < a\} = \cup_{N \geq m} \cap_{n \geq N} \{X_{g_n(\alpha)} < a\} \in \mathcal{F}_{g_m(\alpha)}, \forall m \geq 1$. Hence $\{X_\alpha < a\} \in \cap_m \mathcal{F}_{g_m(\alpha)} = \mathcal{F}^r_\alpha$. □

For the rest of the section we will assume that $X := (X_A)_{A \in \mathcal{A}}$ is a fixed set-indexed process and $(\mathcal{F}_A)_{A \in \mathcal{A}}$ is its minimal filtration. We will denote with $\mathcal{B}(\mathbf{R})$ the class of all Borel subsets of $\mathbf{R}$.

The following σ-fields have been introduced by Ivanoff and Merzbach (2000a) for any random set α:

$$\mathcal{F}^X_\alpha \stackrel{\text{def}}{:=} \sigma(\{X_A I_{\{A \subseteq \alpha\}}, I_{\{A \subseteq \alpha\}}; A \in \mathcal{A}\})$$

$$\mathcal{F}^X_{\partial\alpha} \stackrel{\text{def}}{:=} \sigma(\{X_A I_{\{A \subseteq \alpha, A \not\subseteq \alpha^0\}}, I_{\{A \subseteq \alpha, A \not\subseteq \alpha^0\}}; A \in \mathcal{A}\})$$

$$\mathcal{F}^X_{\alpha^c} \stackrel{\text{def}}{:=} \sigma(\{X_A I_{\{A \not\subseteq \alpha\}}, I_{\{A \not\subseteq \alpha\}}; A \in \mathcal{A}\})$$

Lemma 2.7 *(Lemma 4.1 of Ivanoff and Merzbach, 2000a)* **(a)** *If* $\alpha \equiv B \in \mathcal{A}(u)$ *then* $\mathcal{F}^X_\alpha = \mathcal{F}_B, \mathcal{F}^X_{\partial\alpha} = \mathcal{F}_{\partial B}, \mathcal{F}^X_{\alpha^c} = \mathcal{F}_{B^c}$.

(b) *For any random set* α *we have* $\mathcal{F}^X_{\partial\alpha} \subseteq \mathcal{F}^X_\alpha$.

Lemma 2.8 **(a)** *For any discrete adapted set* α *we have* $\mathcal{F}^X_\alpha = \mathcal{F}_\alpha$.

(b) *For any optional set* α *we have* $\mathcal{F}^r_\alpha = \cap_n \mathcal{F}^X_{g_n(\alpha)}, \mathcal{F}^X_{g_{n+1}(\alpha)} \subseteq \mathcal{F}^X_{g_n(\alpha)}\ \forall n$.

(c) *For any optional set* α *we have* $\mathcal{F}^X_\alpha \subseteq \mathcal{F}^r_\alpha$.

Proof: **(a)** To prove that $\{A \subseteq \alpha\} \in \mathcal{F}_\alpha$, note that $\{A \subseteq \alpha\} \cap \{\alpha = B\}$ is $\emptyset$ if $A \not\subseteq B$ and it is exactly $\{\alpha = B\}$ otherwise; in either case this intersection lies in $\mathcal{F}_B$. Similarly $\{X_A \in \Gamma\} \cap \{A \subseteq \alpha\} \in \mathcal{F}_\alpha$ for every $\Gamma \in \mathcal{B}(\mathbf{R})$. Conversely, let $F \in \mathcal{F}_\alpha$. Using Lemma 4.6 of Ivanoff and Merzbach (2000a), $F \cap \{\alpha = B\} \in \mathcal{F}_B|_{\{\alpha=B\}} = \mathcal{F}^X_\alpha|_{\{\alpha=B\}} \subseteq \mathcal{F}^X_\alpha$ since $\{\alpha = B\} \in \mathcal{F}^X_{\partial\alpha} \subseteq \mathcal{F}^X_\alpha$. Finally $F = \cup_{B \in \text{range}(\alpha)}(F \cap \{\alpha = B\}) \in \mathcal{F}^X_\alpha$.

(b) Using (a), $\mathcal{F}^X_{g_n(\alpha)} = \mathcal{F}_{g_n(\alpha)}$ and hence $\mathcal{F}^r_\alpha = \cap_n \mathcal{F}_{g_n(\alpha)} = \cap_n \mathcal{F}^X_{g_n(\alpha)}$.

(c) For each $A \in \mathcal{A}$ we have $I_{\{A\subseteq\alpha\}} = \lim_n I_{\{A \subseteq g_n(\alpha)\}}$ and $X_A I_{\{A\subseteq\alpha\}} = \lim_n X_A I_{\{A\subseteq g_n(\alpha)\}}$. Since $I_{\{A\subseteq g_n(\alpha)\}}$ and $X_A I_{\{A\subseteq g_n(\alpha)\}}$ are $\mathcal{F}^X_{g_n(\alpha)}$-measurable and the σ-fields $(\mathcal{F}^X_{g_n(\alpha)})_n$ are decreasing, it follows that the limits $I_{\{A\subseteq\alpha\}}$ and $X_A I_{\{A\subseteq\alpha\}}$ are measurable with respect to the intersection $\cap_n \mathcal{F}^X_{g_n(\alpha)} = \mathcal{F}^r_\alpha$. □

Ideally, we would like to say that a sharp Markov process $X := (X_A)_{A\in\mathcal{A}}$ is strong Markov if for any optional set α, we have

$$\mathcal{F}^r_\alpha \perp \mathcal{F}^X_{\alpha^c} \mid \mathcal{F}^X_{\partial\alpha} \tag{5}$$

which would imply that $\mathcal{F}^X_\alpha \perp \mathcal{F}^X_{\alpha^c} \mid \mathcal{F}^X_{\partial\alpha}$, using Lemma 2.8, (c). However, we would rather not introduce a new terminology, since we could not find an example of a process satisfying this property.

In what follows we will see how close we can get of the desired conditional independence (5) for an arbitrary sharp Markov process. The following σ-field has been introduced by Ivanoff and Merzbach (2000a) for any random set α:

$$\mathcal{F}^X_{[\alpha,g_n(\alpha)]} \stackrel{\text{def}}{:=} \sigma(\{X_A 1_{\{A\subseteq g_n(\alpha), A\not\subseteq\alpha^0\}}, 1_{\{A\subseteq g_n(\alpha), A\not\subseteq\alpha^0\}}; A \in \mathcal{A}\})$$

Lemma 2.9 *Suppose that* $g_n(A) \in \mathcal{A}_n,\ \forall A \in \mathcal{A},\ \forall n \geq 1$. *Then for every optional set* α, $\mathcal{F}^X_{\partial g_n(\alpha)} \subseteq \mathcal{F}^X_{[\alpha,g_n(\alpha)]} \subseteq \mathcal{F}^X_{g_n(\alpha)}$.

Proof: The first inclusion is valid for any random set α. To prove it, it will be enough to show that the generators of $\mathcal{F}^X_{\partial g_n(\alpha)}$ are in $\mathcal{F}^X_{[\alpha,g_n(\alpha)]}$. Note that the set $\{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq g_n(\alpha)^0\}$ can be written as the difference

$$(\{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq \alpha^0\}) \setminus (\{A \subseteq g_n(\alpha)^0\} \cap \{A \not\subseteq \alpha^0\})$$

and the first set $\{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq \alpha^0\}$ lies in $\mathcal{F}_{[\alpha,g_n(\alpha)]}$. As for the second set $\{A \subseteq g_n(\alpha)^0\} \cap \{A \not\subseteq \alpha^0\}$, it can be written as

$$\cup_k(\{g_k(A) \subseteq g_n(\alpha)\} \cap \{A \not\subseteq \alpha^0\}) = \cup_k \cap_{m\geq k}(\{g_m(A) \subseteq g_n(\alpha)\} \cap \{g_m(A) \not\subseteq \alpha^0\})$$

which belongs to $\mathcal{F}^X_{[\alpha,g_n(\alpha)]}$ since $g_m(A) \in \mathcal{A}$. A similar argument shows that $\{X_A \in \Gamma\} \cap \{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq g_n(\alpha)^0\} \in \mathcal{F}_{[\alpha,g_n(\alpha)]}$ for any $\Gamma \in \mathcal{B}(\mathbf{R})$.

To prove the second inclusion we will assume that α is an optional set and we will show that the generators of $\mathcal{F}^X_{[\alpha,g_n(\alpha)]}$ are in $\mathcal{F}^X_{g_n(\alpha)}$. Note that $\{A \not\subseteq \alpha^0\} = \cap_k \{g_k(A) \not\subseteq \alpha\} \in \mathcal{F}^X_\alpha \subseteq \mathcal{F}^r_\alpha \subseteq \mathcal{F}^X_{g_n(\alpha)}$ by Lemma 2.8. Hence $\{A \not\subseteq \alpha^0\} \cap \{A \subseteq g_n(\alpha)\} \in \mathcal{F}^X_{g_n(\alpha)}$. Finally, a similar argument shows that $\{X_A \in \Gamma\} \cap \{A \not\subseteq \alpha^0\} \cap \{A \subseteq g_n(\alpha)\} \in \mathcal{F}^X_{g_n(\alpha)}$ for any $\Gamma \in \mathcal{B}(\mathbf{R})$. □

Since the σ-fields $(\mathcal{F}^X_{[\alpha,g_n(\alpha)]})_n$ may not necessarily be monotonically decreasing, we let

$$\mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]} := \bigvee_{N\geq n} \mathcal{F}^X_{[\alpha,g_N(\alpha)]} \text{ and } \overline{\mathcal{F}^X_{\partial\alpha}} := \cap_n \mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}$$

Note that $\mathcal{F}^X_{\partial\alpha} \subseteq \mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}$ since $\{A \subseteq \alpha, A \not\subseteq \alpha^0\} = \cap_{k\geq n}\{A \subseteq g_k(\alpha), A \not\subseteq \alpha^0\} \in \mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}$, and similarly $\{X_A \in \Gamma, A \subseteq \alpha, A \not\subseteq \alpha^0\} \in \mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}$ for any $\Gamma \in \mathcal{B}(\mathbf{R})$; hence $\mathcal{F}^X_{\partial\alpha} \subseteq \overline{\mathcal{F}^X_{\partial\alpha}}$.

By Lemma 2.9, for every optional set α

$$\mathcal{F}^X_{\partial g_n(\alpha)} \subseteq \mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]} \subseteq \mathcal{F}^X_{g_n(\alpha)} \tag{6}$$

Theorem 2.10 *If $X := (X_A)_{A\in\mathcal{A}}$ is a sharp Markov process, then for every optional set α, $\mathcal{F}^r_\alpha \perp \mathcal{F}^X_{\alpha^c} \mid \overline{\mathcal{F}^X_{\partial\alpha}}$.*

Proof: The argument is similar to the one used in the proof of Theorem 4.19 of Ivanoff and Merzbach (2000a). Let Y be a bounded, $\mathcal{F}^X_{\alpha^c}$-measurable random variable. Then $Y = \lim_n Y_n$, where Y_n is a bounded, $\mathcal{F}^X_{g_n(\alpha)^c}$-measurable random variable. Using Theorem 4.7 of Ivanoff and Merzbach (2000a) and (6), $E[Y_n|\mathcal{F}^X_{g_n(\alpha)}] = E[Y_n|\mathcal{F}^X_{\partial g_n(\alpha)}] = E[Y_n|\mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}]$. Since $\mathcal{F}^r_\alpha = \cap_n \mathcal{F}^X_{g_n(\alpha)}$ and $\overline{\mathcal{F}^X_{\partial\alpha}} = \cap_n \mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}$ we get

$$E[Y|\mathcal{F}^r_\alpha] = \lim_n E[Y_n|\mathcal{F}^X_{g_n(\alpha)}] = \lim_n E[Y_n|\mathcal{F}^{X,m}_{[\alpha,g_n(\alpha)]}] = E[Y|\overline{\mathcal{F}^X_{\partial\alpha}}]$$

using a generalized form of the Martingale Convergence Theorem. □

Corollary 2.11 *If $X := (X_A)_{A\in\mathcal{A}}$ is a sharp Markov process, then for every optional set α, $\mathcal{F}^X_\alpha \perp \mathcal{F}^X_{\alpha^c} \mid \overline{\mathcal{F}^X_{\partial\alpha}}$.*

Comment 2.12 The applicability of the preceding results extends to all sharp Markov processes, without any requirement for regularity of sample paths or filtrations. Therefore, this holds for all processes with independent increments, including the Brownian motion on the lower layers, which is known to be a.s. unbounded, and therefore a.s. discontinuous.

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