DLW(2001/01/01)

Superprocesses with Dependent Spatial Motion and General Branching Densities

Donald A. Dawson[1]
School of Mathematics and Statistics, Carleton University,
1125 Colonel By Drive, Ottawa
Ontario K1S 5B6, Canada

Zenghu Li[2]
Department of Mathematics, Beijing Normal University,
Beijing 100875, P.R. China

Hao Wang[3]
Department of Mathematics, University of Oregon,
Eugene OR 97403-1222, U.S.A.

Abstract

We construct a class of superprocesses by taking the high density limit of a sequence of interacting-branching particle systems. The spatial motion of the superprocess is determined by a system of interacting diffusions, the branching density is given by an arbitrary bounded non-negative Borel function, and the superprocess is characterized by a martingale problem as a diffusion process with state space $M(\mathbb{R})$, improving and extending considerably the construction of Wang (1997, 1998). It is then proved in a special case that a suitable rescaled process of the superprocess converges to the usual super Brownian motion. An extension to measure-valued branching catalysts is also discussed.

AMS Subject Classifications: Primary 60J80, 60G57; Secondary 60J35

Key words and phrases: superprocess, interacting-branching particle system, diffusion process, martingale problem, dual process, rescaled limit, measure-valued catalyst.

---

[1]Supported by NSERC operating grant.
[2]Supported by NNSF grant 19361060.
[3]Supported by the research grant of UO.

1 Introduction

For a given topological space E, let $B(E)$ denote the totality of all bounded Borel functions on E and let $C(E)$ denote its subset comprising of continuous functions. Let $M(E)$ denote the space of finite Borel measures on E endowed with the topology of weak convergence. Write $\langle f, \mu\rangle$ for $\int f d\mu$. For $F \in B(M(E))$ let

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{h\to 0^+} \frac{1}{h}[F(\mu + h\delta_x) - F(\mu)], \quad x \in E, \tag{1.1}$$

if the limit exists. Let $(\delta^2 F/\delta\mu(x)\delta\mu(y))(\mu)$ be defined in the same way with F replaced by $(\delta F/\delta\mu(y))$ on the right hand side. For example, if $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ for $f \in B(E^m)$ and $\mu \in M(E)$, then

$$\frac{\delta F_{m,f}}{\delta \mu(x)}(\mu) = \sum_{i=1}^{m} \langle \Psi_i(x) f, \mu^{m-1}\rangle, \quad x \in E, \tag{1.2}$$

where $\Psi_i(x)$ is the operator from $B(E^m)$ to $B(E^{m-1})$ defined by

$$\Psi_i(x) f(x_1, \cdots, x_{m-1}) = f(x_1, \cdots, x_{i-1}, x, x_i, \cdots, x_{m-1}), \quad x_j \in E, \tag{1.3}$$

where $x \in E$ is the ith variable of f on the right hand side.

Now we consider the case where $E = I\!\!R$, the one-dimensional Euclidean space. Suppose that $c \in C(I\!\!R)$ is Lipschitz and $g \in C(I\!\!R)$ is square-integrable. Let

$$\rho(z) = \int_{I\!\!R} g(y-z) g(y) dy, \quad z \in I\!\!R, \tag{1.4}$$

and $a(x) = c(x)^2 + \rho(0)$ for $x \in I\!\!R$. We assume in addition that ρ is twice continuously differentiable with ρ' and ρ'' bounded, which is satisfied if g is integrable and twice continuously differentiable with g' and g'' bounded. Then

$$\begin{aligned} \mathcal{A}F(\mu) &= \frac{1}{2}\int_{I\!\!R} a(x) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ &\quad + \frac{1}{2}\int_{I\!\!R^2} \rho(x-y) \frac{d^2}{dx dy} \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \mu(dx)\mu(dy) \end{aligned} \tag{1.5}$$

defines an operator $\mathcal{A}$ which acts on a subset of $B(M(I\!\!R))$ and generates a diffusion process with state space $M(I\!\!R)$. Suppose that $\{W(x,t) : x \in I\!\!R, t \geq 0\}$ is a Brownian sheet and $\{B_i(t) : t \geq 0\}$, $i = 1, 2, \cdots$, is a family of independent standard Brownian motions which are independent of $\{W(x,t) : x \in I\!\!R, t \geq 0\}$. By Lemma 3.1, for any initial conditions $x_i(0) = x_i$, the stochastic equations

$$dx_i(t) = c(x_i(t)) dB_i(t) + \int_{I\!\!R} g(y - x_i(t)) W(dy, dt), \quad t \geq 0, i = 1, 2, \cdots. \tag{1.6}$$

have unique solutions $\{x_i(t) : t \geq 0\}$ and, for each integer $m \geq 1$, $\{(x_1(t), \cdots, x_m(t)) : t \geq 0\}$ is an m-dimensional diffusion process which is generated by the differential operator

$$G^m := \frac{1}{2}\sum_{i=1}^{m} a(x_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.7}$$

In particular, $\{x_i(t) : t \geq 0\}$ is a one-dimensional diffusion process with generator $G := (a(x)/2)\Delta$. Because of the exchangebility, a diffusion process generated by G^m can be regarded as an interacting particle system or a measure-valued process. Heuristically, $a(\cdot)$ represents the speed of the particles and $\rho(\cdot)$ describes the interaction between them. The diffusion process generated by $\mathcal{A}$ arises as the high density limit of a sequence of interacting particle systems described by (1.6); see Wang (1997, 1998) and section 4 of this paper. For $\sigma \in B(\mathbb{R})^+$, we may also define the operator $\mathcal{B}$ by

$$\mathcal{B}F(\mu) = \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx). \tag{1.8}$$

Let $\mathcal{L} = \mathcal{A} + \mathcal{B}$. A Markov process generated by $\mathcal{L}$ is naturally called a *superprocess with dependent spatial motion (SDSM)* with parameters (a, ρ, σ), where σ represents the branching density of the process. In the special case where both c and σ are constants, the SDSM has been constructed in Wang (1997, 1998) as a diffusion process in $M(\hat{\mathbb{R}})$, where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$ is the one-point compactification of $\mathbb{R}$. It was also assumed in Wang (1997, 1998) that g is a symmetric function and that the initial state of the SDSM has compact support in $\mathbb{R}$. Stochastic partial differential equations and local times associated with the SDSM were studied in Dawson et al (2000a, b).

The SDSM contains as special cases several models arising in different circumstances such as the one-dimensional super Brownian motion, the molecular diffusion with turbulent transport and some interacting diffusion systems of McKean-Vlasov type; see e.g. Chow (1976), Dawson (1994), Dawson and Vaillancourt (1995) and Kotelenez (1992, 1995). It is thus of interest to construct the SDSM under reasonably more general conditions and formulate it as a diffusion processes in $M(\mathbb{R})$. This is the main purpose of the present paper. The rest of this paragraph describes the main results of the paper and gives some unsolved problems in the subject. In section 2, we define some function-valued dual process and investigate its connection to the solution of martingale problem of an SDSM. Duality method plays an important role in the investigation. Although the SDSM could arise as high density limit of a sequence of interacting-branching particle systems with location-dependent killing density σ and binary branching distribution, the construction of such systems seems rather sophisticated and is thus avoided in this work. In section 3, we construct the interacting-branching particle system with uniform killing density and location-dependent branching distribution, which is comparatively easier to treat. The arguments are similar to those in Wang (1998). The high density limit of the interacting-branching particle system is considered in section 4, which gives a solution of the martingale problem of the SDSM in the special case where $\sigma \in C(\mathbb{R})^+$ can be extended into a continuous function on $\hat{\mathbb{R}}$. In section 5, we use the dual process to prove the uniqueness of the solution and extend the construction of the SDSM to a general bounded Borel branching density $\sigma \in B(\mathbb{R})^+$. In both sections 4 and 5, we first construct the SDSM as a diffusion process in $M(\hat{\mathbb{R}})$ and then use martingale arguments to show that, if the process is initially supported by $\mathbb{R}$, it always lives in $M(\mathbb{R})$, which is a new result even in the special case considered in Wang (1997, 1998). In section 6, we prove a rescaled limit theorem of the SDSM, which states that a suitable rescaled SDSM converges to the usual super Brownian motion if $c(\cdot)$ is bounded away from zero. This describes another situation where the super Brownian motion arises universally; see also Durrett and Perkins (1998) and Hara and Slade (2000a, b). When $c(\cdot) \equiv 0$, we expect that the same rescaled limit would lead to a measure-valued diffusion process which is the high density limit of a sequence of coalescing-branching particle systems, but there is still a long way to reach a vigorous proof. It suffices to mention that not only the characterization of those high density limits but also that of the coalescing-branching particle systems themselves are still open problems. We refer the reader to Evans and Pitman (1998) and the references therein for some recent work on related models. In section 7, we consider an extension of the construction of the SDSM to the case where σ is of the form $\sigma = \dot{\eta}$ with η belonging to a large class of Radon measures on $\mathbb{R}$, in the lines of Dawson and Fleischmann (1991, 1992). The process is constructed only when $c(\cdot)$ is bounded away from zero and it can be called a *superprocess with dependent spatial motion and measure-valued catalysts (SDSMMC)*. The transition semigroup of the SDSMMC is constructed and characterized using a measure-valued dual process. The derivation is based on some estimates of moments of the dual process. However, the existence of a diffusion realization of the SDSMMC is left as another open problem in the subject.

Notation: Recall that $\hat{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$ denotes the one-point compactification of $\mathbb{R}$. Let λ^m denote the Lebesgue measure on $\mathbb{R}^m$. Let $C^n_\partial(\mathbb{R}^m)$ be the set of n-times continuously differentiable functions on $\mathbb{R}^m$ which together with their derivatives up to the nth order can be extended to ∂ continuously. Let $C^n_0(\mathbb{R}^m)$ be the subset of $C^n_\partial(\mathbb{R}^m)$ of functions that together with their derivatives up to the nth order vanish at ∂. Let $(P^m_t)_{t\geq 0}$ denote the transition semigroup of the m-dimensional standard Brownian motion and let $(T^m_t)_{t\geq 0}$ denote the transition semigroup generated by the operator G^m. We shall omit the superscripts m and n when they equal one. Let $(\hat{T}_t)_{t\geq 0}$ and $\hat{G}$ denote the extensions of $(T_t)_{t\geq 0}$ and G to $\hat{\mathbb{R}}$ with ∂ as a trap.

We remark that, if $|c(x)| \geq \epsilon > 0$ for all $x \in \mathbb{R}$, the semigroup $(T^m_t)_{t\geq 0}$ has a density $p^m_t(x, y)$ which satisfies

$$p^m_t(x, y) \leq \text{const} \cdot g^m_{\epsilon t}(x, y), \qquad t \geq 0, x, y \in \mathbb{R}^m, \tag{1.9}$$

where $g^m_t(x, y)$ denotes the transition density of the m-dimensional standard Brownian motion; see e.g. Friedman (1964, p.24).

2 Function-valued dual processes

In this section, we define a function-valued dual process and investigate its connection to the solution of the martingale problem for the SDSM. Suppose that $\sigma \in B(I\!\!R)^+$. Observe that, if $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $f \in C^2_\partial(I\!\!R^m)$, then

$$\begin{aligned} \mathcal{A}F_{m,f}(\mu) &= \frac{1}{2}\int_{I\!\!R^m} \sum_{i=1}^m a(x_i)) f''_{ii}(x_1, \cdots, x_m)\mu^m(dx_1, \cdots, dx_m) \\ &\quad + \frac{1}{2}\int_{I\!\!R^m} \sum_{i,j=1, i\neq j}^m \rho(x_i - x_j) f''_{ij}(x_1, \cdots, x_m)\mu^m(dx_1, \cdots, dx_m) \\ &= F_{m,G^m f}(\mu), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \mathcal{B}F_{m,f}(\mu) &= \frac{1}{2}\sum_{i,j=1, i\neq j}^m \int_{I\!\!R^{m-1}} \Phi_{ij} f(x_1, \cdots, x_{m-1})\mu^m(dx_1, \cdots, dx_{m-1}) \\ &= \frac{1}{2}\sum_{i,j=1, i\neq j}^m F_{m-1,\Phi_{ij} f}(\mu), \end{aligned} \tag{2.2}$$

where Φ_{ij} denotes the operator from $B(E^m)$ to $B(E^{m-1})$ defined by

$$\Phi_{ij} f(x_1, \cdots, x_{m-1}) = \sigma(x_{m-1}) f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}), \tag{2.3}$$

where x_{m-1} is in the places of the ith and the jth variables of f on the right hand side. It follows that

$$\mathcal{L}F_{m,f}(\mu) = F_{m,G^m f}(\mu) + \frac{1}{2}\sum_{i,j=1, i\neq j}^m F_{m-1,\Phi_{ij} f}(\mu). \tag{2.4}$$

Let $\{M_t : t \geq 0\}$ be a nonnegative integer-valued pure jump Markov process with transition intensities $q_{i,j}$ such that $q_{i,i-1} = i(i-1)/2$ and $q_{i,j} = 0$ for all other pairs (i, j). Let $\tau_0 = 0$ and $\tau_{M_0} = \infty$, and let $\{\tau_k : 1 \leq k \leq M_0 - 1\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$. Let $\{\Gamma_k : 1 \leq k \leq M_0 - 1\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\boldsymbol{P}\{\Gamma_k = \Phi_{i,j} | M(\tau_k^-) = l\} = \frac{1}{l(l-1)}, \quad 1 \leq i \neq j \leq l, \tag{2.5}$$

where $\Phi_{i,j}$ is defined by (2.3). Let $I\!\!B$ denote the topological union of $\{B(I\!\!R^m) : m = 1, 2, \cdots\}$ endowed with pointwise convergence on each $B(I\!\!R^m)$. Then

$$Y_t = T^{M_{\tau_k}}_{t-\tau_k} \Gamma_k T^{M_{\tau_{k-1}}}_{\tau_k - \tau_{k-1}} \Gamma_{k-1} \cdots T^{M_{\tau_1}}_{\tau_2 - \tau_1} \Gamma_1 T^{M_0}_{\tau_1} Y_0, \quad \tau_k \leq t < \tau_{k+1}, 0 \leq k \leq M_0 - 1, \tag{2.6}$$

defines a Markov process $\{Y_t : t \geq 0\}$ taking values from $I\!B$. Clearly, $\{(M_t, Y_t) : t \geq 0\}$ is also a Markov process. To simplify the presentation, we shall suppress the dependence of $\{Y_t : t \geq 0\}$ on σ and let $\boldsymbol{Q}^\sigma_{m,f}$ denote the expectation given $M_0 = m$ and $Y_0 = f \in C(I\!R^m)$, just as we are working with a canonical realization of $\{(M_t, Y_t) : t \geq 0\}$.

Lemma 2.1 *For any $f \in B(I\!R^m)$ and any integer $m \geq 1$,*

$$\begin{aligned}&\boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ \leq\;& \|f\| \sum_{k=0}^{m-1} 2^{-k} m^k (m-1)^k \|\sigma\|^k \langle 1, \mu\rangle^{m-k} t^k. \end{aligned} \tag{2.7}$$

Proof. For $0 \leq k \leq m-1$ set

$$A_k = \boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\} 1_{\{\tau_k \leq t < \tau_{k+1}\}}\Big].$$

Then $A_0 = \langle T^m_t f, \mu^m\rangle \leq \|f\|\langle 1, \mu\rangle^m$. By (2.6), for $1 \leq k \leq m-1$, A_k is equal to

$$\frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t \boldsymbol{E}\langle T^{m-k}_{t-s_k}\Gamma_k \cdots T^{m-1}_{s_2-s_1}\Gamma_1 T^m_{s_1} f, \mu^{m-k}\rangle ds_k,$$

from which we get

$$\begin{aligned} A_k \;&\leq\; \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t \|f\|\|\sigma\|^k \langle 1, \mu\rangle^{m-k} ds_k \\ &\leq\; 2^{-k} m^k (m-1)^k \|f\|\|\sigma\|^k \langle 1, \mu\rangle^{m-k} t^k. \end{aligned}$$

Then we have the conclusion. □

Lemma 2.2 *Suppose that $\sigma_n \to \sigma$ boundedly and pointwise and $\mu_n \to \mu$ in $M(I\!R)$ as $n \to \infty$. Then, for any $f \in B(I\!R^m)$ and any integer $m \geq 1$,*

$$\begin{aligned}&\boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =\;& \lim_{n\to\infty} \boldsymbol{Q}^{\sigma_n}_{m,f}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \end{aligned} \tag{2.8}$$

Proof. By the construction (2.6) we have

$$\begin{aligned}&\boldsymbol{Q}^{\sigma_n}_{m,f}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =\;& \langle T^m_t f, \mu_n^m\rangle \\ &+\frac{1}{2}\sum_{i,j=1,i\neq j}^m \int_0^t \boldsymbol{Q}^{\sigma_n}_{m-1,\Phi_{ij}T^m_u f}\Big[\langle Y_{t-u}, \mu_n^{M_{t-u}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\}\Big]du. \end{aligned} \tag{2.9}$$

If $h \in C(I\!R^2)$, then

$$\begin{aligned}
& \boldsymbol{Q}^{\sigma_n}_{1,\Phi_{12}h}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= \;& \boldsymbol{Q}^{\sigma_n}_{1,\Phi_{21}h}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= \;& \int_{I\!R^2} h(y,y) p_t(x,y)\mu_n(dx)\sigma_n(y)dy. \qquad (2.10)
\end{aligned}$$

If $f, g \in C(I\!R)^+$ have bounded supports, then we have $f(x)\mu_n(dx) \to f(x)\mu(dx)$ and $g(y)\sigma_n(y)dy \to g(y)\sigma(y)dy$ by weak convergence, so that

$$\lim_{n\to\infty}\int_{I\!R^2} f(x)g(y)p_t(x,y)\mu_n(dx)\sigma_n(y)dy = \int_{I\!R^2} f(x)g(y)p_t(x,y)\mu(dx)\sigma(y)dy.$$

Since $\{\mu_n\}$ is tight and $\{\sigma_n\}$ is bounded, one can easily see that $\{p_t(x,y)\mu_n(dx)\sigma_n(y)dy\}$ is a tight sequence and hence $p_t(x,y)\mu_n(dx)\sigma_n(y)dy \to p_t(x,y)\mu(dx)\sigma(y)dy$ by weak convergence. Therefore, the value of (2.10) converges as $n \to \infty$ to

$$\begin{aligned}
& \boldsymbol{Q}^{\sigma}_{1,\Phi_{12}h}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= \;& \boldsymbol{Q}^{\sigma}_{1,\Phi_{21}h}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= \;& \int_{I\!R^2} h(y,y) p_t(x,y)\mu(dx)\sigma(y)dy.
\end{aligned}$$

Applying bounded convergence theorem to (2.9) we get inductively

$$\begin{aligned}
& \boldsymbol{Q}^{\sigma}_{m-1,\Phi_{ij}T_t^m f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= \;& \lim_{n\to\infty} \boldsymbol{Q}^{\sigma_n}_{m-1,\Phi_{ij}T_t^m f}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]
\end{aligned}$$

for $1 \leq i \neq j \leq m$. Then the result follows from (2.9). □

Theorem 2.1 *Let $\mathcal{D}(\mathcal{L})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C^2_\partial(I\!R^m)$. Suppose that $\{X_t : t \geq 0\}$ is a continuous $M(I\!R)$-valued process such that $\boldsymbol{E}\{\langle 1, X_t\rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$. If $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem, then*

$$\boldsymbol{E}\langle f, X_t^m\rangle = \boldsymbol{Q}^{\sigma}_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \qquad (2.11)$$

for any $t \geq 0$, $f \in B(I\!R^m)$ and integer $m \geq 1$, *Proof.* It suffices to prove the equality for $f \in C^2_\partial(I\!R^m)$. In this proof, we set $F_\mu(m,f) = F_{m,f}(\mu) = \langle f, \mu^m\rangle$. From the construction (2.6), it is not hard to see that $\{(M_t, Y_t) : t \geq 0\}$ is generated by $\mathcal{L}^*$ which is given by

$$\begin{aligned}\mathcal{L}^* F_\mu(m,f) &= F_\mu(m, G^m f) + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} [F_\mu(m-1, \Phi_{ij} f) - F_\mu(m,f)] \\ &= \mathcal{L} F_{m,f}(\mu) - \frac{1}{2} m(m-1) F_{m,f}(\mu). \qquad (2.12)\end{aligned}$$

In the sequel of the proof, we assume that $\{X_t : t \geq 0\}$ and $\{(M_t, Y_t) : t \geq 0\}$ are defined on the same probability space and are independent of each other. For any partition $\Delta_n := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0,t]$, we have

$$\begin{aligned}&\boldsymbol{E}\langle f, X_t^m\rangle - \boldsymbol{E}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \sum_{i=1}^{n}\Big(\boldsymbol{E}\Big[\langle Y_{t-t_i}, X_{t_i}^{M_{t-t_i}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big] \\ &-\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_{i-1}} M_s(M_s-1)ds\Big\}\Big]\Big).\end{aligned}$$

Let $\|\Delta_n\| = \max\{|t_i - t_{i-1}| : 1 \leq i \leq n\}$ and assume $\|\Delta_n\| \to 0$ as $n \to \infty$. By the independence of $\{X_t : t \geq 0\}$ and $\{(M_t, Y_t) : t \geq 0\}$ and the martingale characterization of $\{(M_t, Y_t) : t \geq 0\}$,

$$\begin{aligned}&\lim_{n\to\infty}\sum_{i=1}^{n}\Big(\boldsymbol{E}\Big[\langle Y_{t-t_i}, X_{t_i}^{M_{t-t_i}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big] \\ &\qquad -\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_i}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big]\Big) \\ =& \lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\boldsymbol{E}\Big[F_{X_{t_i}}(M_{t-t_i}, Y_{t-t_i}) \\ &\qquad -F_{X_{t_i}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}})\Big| X; \{(M_r, Y_r) : 0 \leq r \leq t-t_i\}\Big]\Big) \\ =& -\lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\ &\qquad \boldsymbol{E}\Big[\int_{t-t_i}^{t-t_{i-1}} \mathcal{L}^* F_{X_{t_i}}(M_u, Y_u)du\Big| X; \{(M_r, Y_r) : 0 \leq r \leq t-t_i\}\Big]\Big) \\ =& -\lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\int_{t-t_i}^{t-t_{i-1}} \mathcal{L}^* F_{X_{t_i}}(M_u, Y_u)du\Big) \\ =& -\lim_{n\to\infty}\int_0^t \boldsymbol{E}\Big(\sum_{i=1}^{n}\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\mathcal{L}^* F_{X_{t_i}}(M_u, Y_u) 1_{[t-t_i, t-t_{i-1}]}(u)\Big)du \\ =& -\int_0^t \boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\}\mathcal{L}^* F_{X_u}(M_u, Y_u)\Big)du,\end{aligned}$$

where the last step holds by the right continuity of $\{X_t : t \geq 0\}$. Using again the independence and the martingale problem for $\{X_t : t \geq 0\}$,

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^n \Big(\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_i}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big] \\
&\qquad -\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big]\Big) \\
= &\lim_{n\to\infty}\sum_{i=1}^n \boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad \boldsymbol{E}\Big[F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_{t_i}) - F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_{t_{i-1}})\Big| M,Y\Big]\Big) \\
= &\lim_{n\to\infty}\sum_{i=1}^n \boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\boldsymbol{E}\Big[\int_{t_{i-1}}^{t_i} \mathcal{L}F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_u)du\Big|M,Y\Big]\Big) \\
= &\lim_{n\to\infty}\sum_{i=1}^n \boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\int_{t_{i-1}}^{t_i} \mathcal{L}F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_u)du\Big) \\
= &\lim_{n\to\infty}\int_0^t \boldsymbol{E}\Big(\sum_{i=1}^n \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\mathcal{L}F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_u)1_{[t_{i-1},t_i]}(u)\Big)du \\
= &\int_0^t \boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\}\mathcal{L}F_{M_{t-u},Y_{t-u}}(X_u)\Big)du,
\end{aligned}$$

where we have also used the right continuity of $\{(M_t, Y_t) : t \geq 0\}$ for the last step. Finally, observe that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^n \Big(\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big] \\
&\qquad -\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_{i-1}} M_s(M_s-1)ds\Big\}\Big]\Big) \\
= &\lim_{n\to\infty}\sum_{i=1}^n \boldsymbol{E}\Big(F_{X_{t_{i-1}}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}}) \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad \Big[1-\exp\Big\{\frac{1}{2}\int_{t-t_i}^{t-t_{i-1}} M_u(M_u-1)du\Big\}\Big]\Big) \\
= &-\lim_{n\to\infty}\sum_{i=1}^n \boldsymbol{E}\Big(F_{X_{t_{i-1}}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}}) \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad \Big[\frac{1}{2}\int_{t-t_i}^{t-t_{i-1}} M_u(M_u-1)du\Big]\Big) \\
= &-\lim_{n\to\infty}\frac{1}{2}\int_0^t \boldsymbol{E}\Big(\sum_{i=1}^n F_{X_{t_{i-1}}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}}) \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad M_u(M_u-1)1_{[t-t_i,t-t_{i-1}]}(u)\Big)du.
\end{aligned}$$

Since the semigroups $(T_t^m)_{t\geq 0}$ are strongly Feller and strongly continuous, $\{Y_t : t \geq 0\}$ is continuous in the uniform norm in each open interval between two neighboring jumps of $\{M_t : t \geq 0\}$. Using this, the left continuity of $\{X_t : t \geq 0\}$ and dominated convergence, we see that the above value equals

$$-\frac{1}{2}\int_0^t \boldsymbol{E}\Big(F_{X_u}(M_{t-u}, Y_{t-u})\exp\left\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\right\}M_u(M_u-1)\Big)du.$$

Combining those together we get (2.11). □

Theorem 2.2 *Let $\mathcal{D}(\mathcal{L})$ be as in Theorem 2.1 and let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0,\infty), M(I\!\!R))$. Suppose that for each $\mu \in M(I\!\!R)$ there is a probability measure $\boldsymbol{P}_\mu$ on $C([0,\infty), M(I\!\!R))$ such that $\boldsymbol{P}_\mu\{\langle 1, w_t\rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$ and $\{w_t : t \geq 0\}$ under $\boldsymbol{P}_\mu$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem. Then the system $\{\boldsymbol{P}_\mu : \mu \in M(I\!\!R)\}$ defines a diffusion process with transition semigroup $(P_t)_{\geq 0}$ given by*

$$\int_{M(I\!\!R)} \langle f, \nu_t^m\rangle P_t(\mu, d\nu) = \boldsymbol{Q}_{m,f}^\sigma\left[\langle Y_t, \mu^{M_t}\rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\right\}\right]. \tag{2.13}$$

Proof. Since the class $\{F_{m,f}\}$ separates probability measures on $M(I\!\!R)$, the result follows from Theorem 2.1; see e.g. Ethier and Kurtz (1986, p.184). □

3 Interacting-branching particle systems

In this section, we give a formulation of the interacting-branching particle system. We first prove that equations (1.6) have unique solutions. Recall that $c \in C(I\!\!R)$ is Lipschitz, $g \in C(I\!\!R)$ is square-integrable and ρ is twice continuously differentiable with ρ' and ρ'' bounded.

Lemma 3.1 *For any initial conditions $x_i(0) = x_i$, equations (1.6) have unique solutions $\{x_i(t) : t \geq 0\}$ and $\{(x_1(t), \cdots, x_m(t)) : t \geq 0\}$ is an m-dimensional diffusion process with generator G^m defined by (1.7).*

Proof. Fix $T > 0$ and $i \geq 1$ and define $\{x_i^k(t) : t \geq 0\}$ inductively by $x_i^0(t) \equiv x_i(0)$ and

$$x_i^{k+1}(t) = x_i(0) + \int_0^t c(x_i^k(s))dB_i(s) + \int_0^t \int_{I\!\!R} g(y - x_i^k(s))W(dy, ds), \quad t \geq 0.$$

Let $l(c) \geq 0$ be any Lipschitz constant for $c(\cdot)$. Then we have

$$\begin{aligned}
\boldsymbol{E}\Big\{\sup_{0\leq t\leq T}|x_i^{k+1}(t)-x_i^k(t)|^2\Big\} &= \int_0^T \boldsymbol{E}\{|c(x_i^k(t))-c(x_i^{k-1}(t))|^2\}dt \\
&\quad + \int_0^T \boldsymbol{E}\Big\{\int_{I\!R}|g(y-x_i^k(t))-g(y-x_i^{k-1}(t))|^2dy\Big\}dt \\
&\leq l(c)^2\int_0^T \boldsymbol{E}\{|x_i^k(t)-x_i^{k-1}(t)|^2\}dt \\
&\quad +2\int_0^T \boldsymbol{E}\{|\rho(0)-\rho(x_i^k(t)-x_i^{k-1}(t))|\}dt \\
&\leq (l(c)^2+\|\rho''\|)\int_0^T \boldsymbol{E}\{|x_i^k(t)-x_i^{k-1}(t)|^2\}dt.
\end{aligned}$$

Using the above inequality inductively we get

$$\boldsymbol{E}\Big\{\sup_{0\leq t\leq T}|x_i^{k+1}(t)-x_i^k(t)|^2\Big\} \leq (\|c\|^2+\rho(0))(l(c)^2+\|\rho''\|)^kT^k/k!,$$

and hence

$$\boldsymbol{E}\Big\{\sup_{0\leq t\leq T}|x_i^{k+1}(t)-x_i^k(t)|>2^{-k}\Big\} \leq \mathrm{const}\cdot(2T)^k(l(c)^2+\|\rho''\|)^k/k!.$$

By Borel-Cantelli's lemma, $\{x_i^k(t): 0\leq t\leq T\}$ converges in the uniform norm with probability one. Since $T>0$ was arbitrary, $x_i(t)=\lim_{k\to\infty}x_i^k(t)$ defines a continuous martingale $\{x_i(t): t\geq 0\}$ which is clearly the unique solution of (1.6). It is easy to see that $d\langle x_i\rangle(t)=a(x_i(t))dt$ and $d\langle x_i,x_j\rangle(t)=\rho(x_i(t)-x_j(t))dt$ for $i\neq j$. Then $\{(x_1(t),\cdots,x_m(t)): t\geq 0\}$ is a diffusion process with generator G^m defined by (1.7). □

Because of the exchangebility, the G^m-diffusion can be regarded as a measure-valued Markov process. Let $N(I\!R)$ denote the space of integer-valued measures on $I\!R$. For $\theta>0$, let $M_\theta(I\!R)=\{\theta^{-1}\sigma:\sigma\in N(I\!R)\}$. Let ζ be the mapping from $\cup_{m=1}^\infty I\!R^m$ to $M_\theta(I\!R)$ defined by

$$\zeta(x_1,\cdots,x_m)=\frac{1}{\theta}\sum_{i=1}^m\delta_{x_i}, \qquad m\geq 1. \tag{3.1}$$

Lemma 3.2 *For any integers $m,n\geq 1$ and any $f\in C^2(I\!R^n)$, we have*

$$\begin{aligned}
G^mF_{n,f}(\zeta(x_1,\cdots,x_m)) &= \frac{1}{2\theta^n}\sum_{\alpha=1}^n\sum_{l_1,\cdots,l_n=1}^m a(x_{l_\alpha})f''_{\alpha\alpha}(x_{l_1},\cdots,x_{l_n}) \\
&+ \frac{1}{2\theta^n}\sum_{\alpha,\beta=1,\alpha\neq\beta}^n\sum_{l_1,\cdots,l_n=1,l_\alpha=l_\beta}^m c(x_{l_\alpha})c(x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}) \\
&+ \frac{1}{2\theta^n}\sum_{\alpha,\beta=1,\alpha\neq\beta}^n\sum_{l_1,\cdots,l_n=1}^m \rho(x_{l_\alpha}-x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned}\tag{3.2}$$

Proof. By (3.1), we have

$$F_{n,f}(\zeta(x_1,\cdots,x_m)) = \frac{1}{\theta^n}\sum_{l_1,\cdots,l_n=1}^{m} f(x_{l_1},\cdots,x_{l_n}). \tag{3.3}$$

Observe that, for $1\le i\le m$,

$$\frac{d^2}{dx_i^2}F_{n,f}(\zeta(x_1,\cdots,x_m)) = \frac{1}{\theta^n}\sum_{\alpha,\beta=1}^{n}\sum_{\{\cdots\}} f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}),$$

where $\{\cdots\} = \{\text{for all } 1\le l_1,\cdots,l_n\le m \text{ with } l_\alpha = l_\beta = i\}$. Then it is not hard to see that

$$\begin{aligned}
&\sum_{i=1}^{m} c(x_i)^2\frac{d^2}{dx_i^2}F_{n,f}(\zeta(x_1,\cdots,x_m))\\
=\ & \frac{1}{\theta^n}\sum_{\alpha,\beta=1}^{n}\sum_{l_1,\cdots,l_n=1,l_\alpha=l_\beta}^{m} c(x_{l_\alpha})c(x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n})\\
=\ & \frac{1}{\theta^n}\sum_{\alpha=1}^{n}\sum_{l_1,\cdots,l_n=1}^{m} c(x_{l_\alpha})^2 f''_{\alpha\alpha}(x_{l_1},\cdots,x_{l_n})\\
& +\frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{l_1,\cdots,l_n=1,l_\alpha=l_\beta}^{m} c(x_{l_\alpha})c(x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned} \tag{3.4}$$

On the other hand, for $1\le i\ne j\le m$,

$$\left(\frac{d^2}{dx_idx_j}+\frac{d^2}{dx_idx_j}\right)F_{n,f}(\zeta(x_1,\cdots,x_m)) = \frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{\{\cdots\}} f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}),$$

where $\{\cdots\} = \{\text{for all } 1\le l_1,\cdots,l_n\le m \text{ with } l_\alpha = i \text{ and } l_\beta = j\}$. It follows that

$$\begin{aligned}
&\sum_{i,j=1,i\ne j}^{m} \rho(x_i-x_j)\frac{d^2}{dx_idx_j}F_{n,f}(\zeta(x_1,\cdots,x_m))\\
=\ & \frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{l_1,\cdots,l_n=1,l_\alpha\ne l_\beta}^{m} \rho(x_{l_\alpha}-x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned}$$

Using this and (3.4) with $c(x_i)^2$ replaced by $\rho(0)$,

$$\begin{aligned}
&\sum_{i,j=1}^{m} \rho(x_i-x_j)\frac{d^2}{dx_idx_j}F_{n,f}(\zeta(x_1,\cdots,x_m))\\
=\ & \frac{1}{\theta^n}\sum_{\alpha=1}^{n}\sum_{l_1,\cdots,l_n=1}^{m} \rho(0)f''_{\alpha\alpha}(x_{l_1},\cdots,x_{l_n})\\
& +\frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{l_1,\cdots,l_n=1}^{m} \rho(x_{l_\alpha}-x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned} \tag{3.5}$$

Then we have the desired result from (3.4) and (3.5). $\square$

Suppose that $X(t) = (x_1(t), \cdots, x_m(t))$ is a Markov process in $I\!R^m$ generated by G^m. Based on (1.2) and Lemma 3.2, it is easy to show that $\zeta(X(t))$ is a Markov process in $M_\theta(I\!R)$ with generator $\mathcal{A}_\theta$ given by

$$\begin{aligned}\mathcal{A}_\theta F(\mu) &= \frac{1}{2}\int_{I\!R} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) + \frac{1}{2\theta}\int_{I\!R^2} c(x)c(y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx) \\ &\quad + \frac{1}{2}\int_{I\!R^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy). \end{aligned} \tag{3.6}$$

In particular, if

$$F(\mu) = f(\langle\phi_1,\mu\rangle, \cdots, \langle\phi_n,\mu\rangle), \quad \mu\in M_\theta(I\!R), \tag{3.7}$$

for $f \in C_0^\infty(I\!R^n)$ and $\phi_i \in C^2(I\!R)$, then

$$\begin{aligned}\mathcal{A}_\theta F(\mu) &= \frac{1}{2}\sum_{i=1}^n f_i'(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi_i'',\mu\rangle \\ &+ \frac{1}{2\theta}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle c^2\phi_i'\phi_j',\mu\rangle \\ &+ \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{I\!R^2}\rho(x-y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy). \end{aligned} \tag{3.8}$$

Now we introduce a branching mechanism to the interacting particle system. Suppose that for each $x \in I\!R$ we have a discrete probability distribution $p = \{p_i(x) : i = 0, 1, \cdots\}$ such that each $p_i(\cdot)$ is a Borel measurable function on $I\!R$. We assume in addition that

$$p_1(x) = 0, \quad \sum_{i=1}^\infty i p_i(x) = 1, \tag{3.9}$$

and

$$\sigma_p(x) := \sum_{i=1}^\infty i^2 p_i(x) - 1 \tag{3.10}$$

is bounded in $x \in I\!R$. Let $\varGamma_\theta(\mu, d\nu)$ be the probability kernel on $M_\theta(I\!R)$ defined by

$$\int_{M_\theta(I\!R)} F(\nu)\varGamma_\theta(\mu,d\nu) = \frac{1}{\theta\mu(1)}\sum_{i=1}^{\theta\mu(1)}\sum_{j=0}^\infty p_j(x_i)F\Big(\mu + (j-1)\theta^{-1}\delta_{x_i}\Big), \tag{3.11}$$

where $\mu \in M_\theta(I\!R)$ is given by

$$\mu = \frac{1}{\theta}\sum_{i=1}^{\theta\mu(1)}\delta_{x_i}.$$

For a constant $\gamma > 0$, we define the bounded operator $\mathcal{B}_\theta$ on $B(M_\theta(I\!\!R))$ by

$$\mathcal{B}_\theta F(\mu) = (\mu(1) \wedge \theta)\theta^2\gamma \int_{M_\theta(I\!\!R)} (F(\nu) - F(\mu))\Gamma_\theta(\mu, d\nu). \tag{3.12}$$

In view of (1.6), $\mathcal{A}_\theta$ generates a Feller Markov process on $M_\theta(I\!\!R)$, then so does $\mathcal{L}_\theta := \mathcal{A}_\theta + \mathcal{B}_\theta$ by Ethier-Kurtz (1986, p.37). We shall call the process generated by $\mathcal{L}_\theta$ an *interacting-branching particle system* with parameters (a, ρ, γ, p) and unit mass $1/\theta$. Heuristically, $a(\cdot)$ represents the speed of the spatial motion, $\rho(\cdot)$ describes the interaction between the particles, $\gamma(\cdot)$ is the branching rate and $p = \{p_i(\cdot) : i = 0, 1, \cdots\}$ gives the distribution of the offspring number. Note that we have added the factor $\wedge\theta$ into the definition (3.12) of $\mathcal{B}_\theta$ to make the branching not too fast even when the number of particles is large. If F is given by (3.7), then $\mathcal{B}_\theta F(\mu)$ equals

$$\frac{(\mu(1) \wedge \theta)\gamma}{2\mu(1)} \sum_{\alpha,\beta=1}^{n} \sum_{j=1}^{\infty} (j-1)^2 \langle p_j f''_{\alpha\beta}(\langle\phi_1, \mu\rangle + \xi_j\phi_1, \cdots, \langle\phi_n, \mu\rangle + \xi_j\phi_n)\phi_\alpha\phi_\beta, \mu\rangle \tag{3.13}$$

for some constant $0 < \xi_j < (j-1)/\theta$. This follows from (3.11) and (3.12) by Taylor's expansion.

4 Continuous branching density

In this section, we shall construct a solution of the martingale problem of the SDSM with continuous branching density by using the particle system approximation. Assume that $\sigma \in C(I\!\!R)$ can be extended continuously to $\hat{I\!\!R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be given by (1.5) and (1.8), respectively. Observe that, if

$$F(\mu) = f(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle), \quad \mu \in M(I\!\!R), \tag{4.1}$$

for $f \in C_0^\infty(I\!\!R^n)$ and $\phi_i \in C_\partial^2(I\!\!R)$, then

$$\begin{aligned} \mathcal{A}F(\mu) &= \frac{1}{2}\sum_{i=1}^{n} f'_i(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)\langle a\phi''_i, \mu\rangle \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} f''_{ij}(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle) \int_{I\!\!R^2} \rho(x-y)\phi'_i(x)\phi'_j(y)\mu(dx)\mu(dy), \end{aligned} \tag{4.2}$$

and

$$\mathcal{B}F(\mu) = \frac{1}{2}\sum_{i,j=1}^{n} f''_{ij}(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)\langle\sigma\phi_i\phi_j, \mu\rangle. \tag{4.3}$$

Let $\{\theta_k\}$ be any sequence such that $\theta_k \to \infty$ as $k \to \infty$. Suppose that $\{X_t^{(k)} : t \geq 0\}$ is a sequence of cadlag interacting-branching particle systems with parameters $(a, \rho, \gamma_k, p^{(k)})$, unit mass $1/\theta_k$ and initial states $X_0^{(k)} = \mu_k \in M_{\theta_k}(I\!R)$. In an obvious way, we may also regard $\{X_t^{(k)} : t \geq 0\}$ as a process with state space $M(\hat{I\!R})$. Let σ_k be defined by (3.10) with p_i replaced by $p_i^{(k)}$.

Lemma 4.1 *Suppose that the sequences $\{\gamma_k\sigma_k\}$ and $\{\langle 1, \mu_k\rangle\}$ are bounded. Then $\{X_t^{(k)} : t \geq 0\}$ form a tight sequence in $D([0,\infty), M(\hat{I\!R}))$.*

Proof. By the assumption (3.9), it is easy to show that $\{\langle 1, X_t^{(k)}\rangle : t \geq 0\}$ is a martingale. Then we have

$$\boldsymbol{P}\Big\{\sup_{t\geq 0}\langle 1, X_t^{(k)}\rangle > \eta\Big\} \leq \frac{\langle 1, \mu_k\rangle}{\eta}$$

for any $\eta > 0$. That is, $\{X_t^{(k)} : t \geq 0\}$ satisfies the compact containment condition of Ethier and Kurtz (1986, p.142). Let $\mathcal{L}_k$ denote the generator of $\{X_t^{(k)} : t \geq 0\}$ and let F be given by (4.1) with $f \in C_0^\infty(I\!R^n)$ and with each $\phi_i \in C_\partial^2(I\!R)$ bounded away from zero. Then

$$F(X_t^{(k)}) - F(X_0^{(k)}) - \int_0^t \mathcal{L}_k F(X_s^{(k)})ds, \quad t \geq 0,$$

is a martingale and the desired tightness follows from the result of Ethier and Kurtz (1986, p.145). □

Now we suppose that all functions in $C_\partial^2(I\!R)$ and their derivatives up to the second order have been extended to $\hat{I\!R}$ by continuity. Then (4.1), (4.2) and (4.3) may also be regarded as functions on $M(\hat{I\!R})$. Let $\hat{\mathcal{A}}F(\mu)$ and $\hat{\mathcal{B}}F(\mu)$ be defined by the right hand side of (4.2) and (4.3), respectively, and let $\hat{\mathcal{L}}F(\mu) = \hat{\mathcal{A}}F(\mu) + \hat{\mathcal{B}}F(\mu)$, viewed as functions on $M(\hat{I\!R})$.

Lemma 4.2 *Let $\mathcal{D}(\hat{\mathcal{L}})$ be the totality of all functions of the form (4.1) with $f \in C_0^\infty(I\!R^n)$ and with each $\phi_i \in C_\partial^2(I\!R)$ bounded away from zero. Suppose further that $\gamma_k\sigma_k \to \sigma$ uniformly and $\mu_k \to \mu \in M(\hat{I\!R})$ as $k \to \infty$. Then any limit point $\boldsymbol{P}_\mu$ of the distributions of $\{X_t^{(k)} : t \geq 0\}$ is supported by $C([0,\infty), M(\hat{I\!R}))$ and is a solution of the $(\hat{\mathcal{L}}, \mathcal{D}(\hat{\mathcal{L}}), \mu)$-martingale problem.*

Proof. We use the notation introduced in the proof of Lemma 4.1. By passing to a subsequence if it is necessary, we may assume that the distribution of $\{X_t^{(k)} : t \geq 0\}$ on $D([0,\infty), M(\hat{I\!R}))$ converges to $\boldsymbol{P}_\mu$. Using Skorokhod's representation, we may assume that the processes $\{X_t^{(k)} : t \geq 0\}$ are defined on the same probability space and the sequence converges almost surely to a cadlag process $\{X_t : t \geq 0\}$ with distribution $\boldsymbol{P}_\mu$

on $D([0,\infty), M(\hat{\mathbb{R}}))$; see e.g. Ethier and Kurtz (1986, p.102). Let $K(X) = \{t \geq 0 : \boldsymbol{P}\{X_t = X_{t-}\} = 1\}$. By Ethier and Kurtz (1986, p.118), for each $t \in K(X)$ we have a.s.

$$\lim_{k\to\infty} X_t^{(k)} = X_t.$$

Recall that f and f''_{ij} are rapidly decreasing and each ϕ_i is bounded away from zero. Since $\gamma_k a_k \to \sigma$ uniformly, for $t \in K(X)$ we have a.s.

$$\lim_{k\to\infty} \mathcal{L}_k F(X_t^{(k)}) = \hat{\mathcal{L}} F(X_t)$$

boundedly by (3.8) and (3.13). Suppose that $\{H_i\}_{i=1}^n \subset C(M(\hat{\mathbb{R}}))$ and $\{t_i\}_{i=1}^{n+1} \subset K(X)$ with $0 \leq t_1 < \cdots < t_n < t_{n+1}$. By Ethier and Kurtz (1986, p.31), the set $K(X)$ is at most countable. Then

$$\begin{aligned}
& \boldsymbol{E}\Big\{\Big[F(X_{t_{n+1}}) - F(X_{t_n}) - \int_{t_n}^{t_{n+1}} \hat{\mathcal{L}} F(X_s) ds\Big] \prod_{i=1}^n H_i(X_{t_i})\Big\} \\
= \ & \boldsymbol{E}\Big\{F(X_{t_{n+1}}) \prod_{i=1}^n H_i(X_{t_i})\Big\} - \boldsymbol{E}\Big\{F(X_{t_n}) \prod_{i=1}^n H_i(X_{t_i})\Big\} \\
& \qquad - \int_{t_n}^{t_{n+1}} \boldsymbol{E}\Big\{\hat{\mathcal{L}} F(X_s) \prod_{i=1}^n H_i(X_{t_i})\Big\} ds \\
= \ & \lim_{k\to\infty} \boldsymbol{E}\Big\{F(X_{t_{n+1}}^{(k)}) \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} - \lim_{k\to\infty} \boldsymbol{E}\Big\{F(X_{t_n}^{(k)}) \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} \\
& \qquad - \lim_{k\to\infty} \int_{t_n}^{t_{n+1}} \boldsymbol{E}\Big\{\mathcal{L}_k F(X_s^{(k)}) \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} ds \\
= \ & \lim_{k\to\infty} \boldsymbol{E}\Big\{\Big[F(X_{t_{n+1}}^{(k)}) - F(X_{t_n}^{(k)}) - \int_{t_n}^{t_{n+1}} \mathcal{L}_k F(X_s^{(k)}) ds\Big] \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} \\
= \ & 0.
\end{aligned}$$

By the right continuity of $\{X_t : t \geq 0\}$ the equality

$$\boldsymbol{E}\Big\{\Big[F(X_{t_{n+1}}) - F(X_{t_n}) - \int_{t_n}^{t_{n+1}} \hat{\mathcal{L}} F(X_s) ds\Big] \prod_{i=1}^n H_i(X_{t_i})\Big\} = 0$$

holds without the restriction $\{t_i\}_{i=1}^{n+1} \subset K(X)$. That is

$$F(X_t) - F(X_0) - \int_0^t \hat{\mathcal{L}} F(X_s) ds, \quad t \geq 0,$$

is a martingale. As in Wang (1998, pp.783-784) one can show that $\{X_t : t \geq 0\}$ is in fact a.s. continuous. □

Lemma 4.3 *Let $\mathcal{D}(\hat{\mathcal{L}})$ be as in Lemma 4.2. Then for each $\mu \in M(\hat{I\!R})$, there is a probability measure $\boldsymbol{P}_\mu$ on $C([0,\infty), M(\hat{I\!R}))$ which is a solution of the $(\hat{\mathcal{L}}, \mathcal{D}(\hat{\mathcal{L}}), \mu)$-martingale problem.*

Proof. It is easy to find $\mu_k \in M_{\theta_k}(I\!R)$ such that $\mu_k \to \mu$ as $k \to \infty$. Then, by Lemma 4.2, it suffices to construct a sequence $(\gamma_k, p^{(k)})$ such that $\gamma_k \sigma_k \to \sigma$ as $k \to \infty$. This is elementary. One choice is described as follows. Let $\gamma_k = 1/\sqrt{k}$ and $\sigma_k = \sqrt{k}(\sigma + 1/\sqrt{k})$. Then the system of equations

$$\begin{cases} p_0^{(k)} + p_2^{(k)} + p_k^{(k)} & = 1, \\ 2p_2^{(k)} + kp_k^{(k)} & = 1, \\ 4p_2^{(k)} + k^2 p_k^{(k)} & = \sigma_k + 1, \end{cases}$$

has the unique solution

$$p_0^{(k)} = \frac{\sigma_k + k - 1}{2k}, \quad p_2^{(k)} = \frac{k - 1 - \sigma_k}{2(k-2)}, \quad p_k^{(k)} = \frac{\sigma_k - 1}{k(k-2)},$$

where $p_i^{(k)}$ is nonnegative for sufficiently large $k \geq 3$. □

Lemma 4.4 *Let $\boldsymbol{P}_\mu$ be given by Lemma 4.3 and let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0,\infty), M(\hat{I\!R}))$. For $n \geq 1$, $t \geq 0$ and $\mu \in M(I\!R)$ we have*

$$\boldsymbol{P}_\mu\{\langle 1, w_t\rangle^n\} \leq \langle 1, \mu\rangle^n + \frac{1}{2}n(n-1)\|\sigma\| \int_0^t \boldsymbol{P}_\mu\{\langle 1, w_s\rangle^{n-1}\}ds.$$

Therefore, $\boldsymbol{P}_\mu\{\langle 1, w_t\rangle^n\}$ is a locally bounded function of $t \geq 0$ and $\boldsymbol{P}_\mu$ is also a solution of the $(\hat{\mathcal{L}}, \mathcal{D}(\hat{\mathcal{L}}), \mu)$-martingale problem if $\mathcal{D}(\hat{\mathcal{L}})$ is the union of all functions of the form (4.1) with $f \in C_0^\infty(I\!R^n)$ and $\phi_i \in C_\partial^2(I\!R)$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C_\partial^2(I\!R^m)$.

Proof. For any $k \geq 1$, take $f_k \in C_0^\infty(I\!R))$ such that $f_k(z) = z^n$ for $0 \leq z \leq k$ and $f_k''(z) \leq n(n-1)z^{n-2}$ for all $z \geq 0$. Let $F_k(\mu) = f_k(\langle 1, \mu\rangle)$. Then $\mathcal{A}F_n(\mu) = 0$ and

$$\mathcal{B}F_k(\mu) \leq \frac{1}{2}n(n-1)\|\sigma\|\langle 1, \mu\rangle^{n-1}.$$

Since

$$F_k(X_t) - F_k(X_0) - \int_0^t \mathcal{L}F_k(\langle 1, X_s\rangle)ds, \quad t \geq 0,$$

is a martingale, we get

$$\begin{aligned} \boldsymbol{P}_\mu f_k(\langle 1, X_t\rangle^n) &\leq f_k(\langle 1, \mu\rangle) + \frac{1}{2}n(n-1)\|\sigma\| \int_0^t \boldsymbol{P}_\mu(\langle 1, X_s\rangle^{n-1})ds \\ &\leq \langle 1, \mu\rangle^n + \frac{1}{2}n(n-1)\|\sigma\| \int_0^t \boldsymbol{P}_\mu(\langle 1, X_s\rangle^{n-1})ds. \end{aligned}$$

Then the desired estimate follows by Fatou's Lemma. □

Lemma 4.5 *Let $\boldsymbol{P}_\mu$ be given by Lemma 4.3. Then for $\mu \in M(\mathbb{R})$ and $\phi \in C^2_\partial(\mathbb{R})$,*

$$M_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \int_0^t \langle a\phi'', w_s \rangle ds, \quad t \geq 0, \tag{4.4}$$

is a $\boldsymbol{P}_\mu$-martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, w_s \rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle g(z - \cdot)\phi', w_s \rangle^2 dz. \tag{4.5}$$

Proof. It is easy to check that, if $F_n(\mu) = \langle \phi, \mu \rangle^n$, then

$$\begin{aligned} \hat{\mathcal{L}} F_n(\mu) \quad &= \quad \frac{n}{2} \langle \phi, \mu \rangle^{n-1} \langle a\phi'', \mu \rangle + \frac{n(n-1)}{2} \langle \phi, \mu \rangle^{n-2} \int_{\hat{\mathbb{R}}} \langle g(z - \cdot)\phi', \mu \rangle^2 dz \\ &\quad + \frac{n(n-1)}{2} \langle \phi, \mu \rangle^{n-2} \langle \sigma\phi^2, \mu \rangle. \end{aligned}$$

It follows that both (4.4) and

$$\begin{aligned} M^2_t(\phi) \quad := \quad & \langle \phi, w_t \rangle^2 - \langle \phi, \mu \rangle^2 - \int_0^t \langle \phi, w_s \rangle \langle a\phi'', w_s \rangle ds \\ & - \int_0^t ds \int_{\hat{\mathbb{R}}} \langle g(z - \cdot)\phi', w_s \rangle^2 dz - \int_0^t \langle \sigma\phi^2, w_s \rangle ds \end{aligned} \tag{4.6}$$

are martingales. By (4.4) and Itô's formula we have

$$\langle \phi, w_t \rangle^2 = \langle \phi, \mu \rangle^2 + \int_0^t \langle \phi, w_s \rangle \langle a\phi'', w_s \rangle ds + 2 \int_0^t \langle \phi, w_s \rangle dM_s(\phi) + \langle M(\phi) \rangle_t. \tag{4.7}$$

Comparing (4.6) and (4.7) we get the conclusion. □

Observe that the martingales $\{M(\phi) : t \geq 0\}$ defined by (4.4) form a system which is linear in $\phi \in C^2_\partial(\mathbb{R})$. Because of the presence of the derivative ϕ' in the variation process (4.5), it seems hard to extend the definition of $\{M(\phi) : t \geq 0\}$ to a general function $\phi \in B(\hat{\mathbb{R}})$. However, following the method of Walsh (1986), one can still define the stochastic integral

$$\int_0^t \int_{\hat{\mathbb{R}}} \phi(s, x) M(ds, dx), \qquad t \geq 0,$$

if both $\phi(s, x)$ and $\phi'(s, x)$ can be extended continuously to $[0, \infty) \times \hat{\mathbb{R}}$. With those in hand, we have the following

Lemma 4.6 *Let $\boldsymbol{P}_\mu$ be given by Lemma 4.3. Then for any $t \geq 0$ and $\phi \in C^2_\partial(\mathbb{R})$ we have a.s.*

$$\langle \phi, w_t \rangle = \langle \hat{T}_t \phi, \mu \rangle + \int_0^t \int_{\hat{\mathbb{R}}} \hat{T}_{t-s} \phi(x) M(ds, dx).$$

Proof. For any partition $\Delta_n := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0,t]$, we have

$$\begin{aligned}\langle \phi, w_t\rangle - \langle \hat{T}_t\phi, \mu\rangle &= \sum_{i=1}^n \langle \hat{T}_{t-t_i}\phi - \hat{T}_{t-t_{i-1}}\phi, w_{t_i}\rangle \\ &\quad + \sum_{i=1}^n [\langle \hat{T}_{t-t_{i-1}}\phi, w_{t_i}\rangle - \langle \hat{T}_{t-t_{i-1}}\phi, w_{t_{i-1}}\rangle].\end{aligned}$$

Let $\|\Delta_n\| = \max\{|t_i - t_{i-1}| : 1 \leq i \leq n\}$ and assume $\|\Delta_n\| \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^n \langle \hat{T}_{t-t_i}\phi - \hat{T}_{t-t_{i-1}}\phi, w_{t_i}\rangle &= -\lim_{n\to\infty} \sum_{i=1}^n \int_{t-t_i}^{t-t_{i-1}} \langle \hat{T}_s \hat{T}_{t-t_i} \hat{G}\phi, w_{t_i}\rangle ds \\ &= -\int_0^t \langle \hat{T}_s \hat{G}\phi, w_s\rangle ds.\end{aligned}$$

Using Lemma 4.5 we have

$$\begin{aligned}&\lim_{n\to\infty} \sum_{i=1}^n [\langle \hat{T}_{t-t_{i-1}}\phi, w_{t_i}\rangle - \langle \hat{T}_{t-t_{i-1}}\phi, w_{t_{i-1}}\rangle] \\ =\ &\lim_{n\to\infty} \sum_{i=1}^n \int_{t-t_i}^{t-t_{i-1}} \int_{\hat{I\!R}} \hat{T}_{t-t_{i-1}}\phi M(ds,dx) + \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^n \int_{t-t_i}^{t-t_{i-1}} \langle a(\hat{T}_{t-t_{i-1}}\phi)'', w_s\rangle ds \\ =\ &\int_0^t \int_{\hat{I\!R}} \hat{T}_{t-s}\phi M(ds,dx) + \frac{1}{2}\int_0^t \langle a(\hat{T}_{t-s}\phi)'', w_s\rangle ds.\end{aligned}$$

Combining those we get the desired conclusion. □

Theorem 4.1 *Let $\mathcal{D}(\mathcal{L})$ be the union of all functions of the form (4.1) with $f \in C_0^\infty(I\!R^n)$ and $\phi_i \in C^2(I\!R)$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C_\partial^2(I\!R^m)$. Then for each $\mu \in M(I\!R)$ there is a probability measure $\boldsymbol{P}_\mu$ on $C([0,\infty), M(I\!R))$ which is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem.*

Proof. Let $\boldsymbol{P}_\mu$ be provided by Lemma 4.3. The desired result will follow once it is proved that

$$\boldsymbol{P}_\mu\{w_t(\{\partial\}) = 0 \text{ for all } t \in [0,u]\} = 1 \tag{4.8}$$

for every $u > 0$. For any $\phi \in C_\partial^2(I\!R)$, we may use Lemma 4.5 to see that

$$M_t^u(\phi) := \langle \hat{T}_{u-t}\phi, w_t\rangle - \langle \hat{T}_u\phi, \mu\rangle = \int_0^t \int_{\hat{I\!R}} \hat{T}_{u-s}\phi M(ds,dx), \quad t \in [0,u],$$

is a continuous martingale with quadratic variation process

$$\langle M^u(\phi)\rangle_t = \int_0^t \langle \sigma(\hat{T}_{u-s}\phi)^2, w_s\rangle ds + \int_0^t ds \int_{\hat{I\!R}} \langle g(z-\cdot)\hat{T}_{u-s}(\phi'), w_s\rangle^2 dz, \quad t \in [0,u].$$

By martingale inequalities, we have

$$\begin{aligned}
& \boldsymbol{P}_\mu\Big\{\sup_{0\le t\le u}|\langle \hat{T}_{u-t}\phi, w_t\rangle - \langle \hat{T}_u\phi,\mu\rangle|^2\Big\} \\
\le\ & 4\int_0^u \boldsymbol{P}_\mu\{\langle\sigma(\hat{T}_{u-s}\phi)^2, w_s\rangle\}ds + 4\int_0^u ds\int_{\hat{I\!R}}\boldsymbol{P}_\mu\{\langle g(z-\cdot)\hat{T}_{u-s}(\phi'), w_s\rangle^2\}dz \\
\le\ & 4\int_0^u \langle\sigma(\hat{T}_{u-s}\phi)^2, \mu\hat{T}_s\rangle ds + 4\int_{\hat{I\!R}} g(z)^2dz\int_0^u \boldsymbol{P}_\mu\{\langle 1, w_s\rangle\langle\hat{T}_{u-s}(\phi')^2, w_s\rangle\}ds \\
\le\ & 4\int_0^u \langle\sigma(\hat{T}_{u-s}\phi)^2, \mu\hat{T}_s\rangle ds + 4\|\phi'\|^2\int_{\hat{I\!R}} g(z)^2dz\int_0^u \boldsymbol{P}_\mu\{\langle 1, w_s\rangle^2\}ds.
\end{aligned}$$

Choose a sequence $\{\phi_k\}\subset C^2_\partial(I\!R)$ such that $\phi_k(\cdot)\to 1_{\{\partial\}}(\cdot)$ boundedly and $\|\phi_k'\|\to 0$ as $k\to\infty$. Replacing ϕ by ϕ_k in the above and letting $k\to\infty$ we obtain (4.8). □

5 Borel branching density

In this section, we shall use the dual process to extend the construction of the SDSM to a general bounded Borel branching density. Given $\sigma\in B(I\!R)^+$, let $\{(M_t,Y_t):t\ge 0\}$ be defined as in section 2. Choose any sequence of functions $\{\sigma_k\}\subset C(I\!R)^+$ which can be extended continuously to $\hat{I\!R}$ and $\sigma_k\to\sigma$ boundedly and pointwise. Let $\{X_t^{(k)}:t\ge 0\}$ be an SDSM with parameters (a,ρ,σ_k) and initial state $\mu_k\in M(I\!R)$.

Theorem 5.1 *Suppose that $\mu_k\to\mu$ in $M(I\!R)$ as $k\to\infty$. Then the distribution of $X_t^{(k)}$ converges to a probability measure $P_t(\mu,\cdot)$ on $M(I\!R)$ determined by*

$$\int_{M(I\!R)}\langle f,\nu^m\rangle P_t(\mu,d\nu) = \boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_t,\mu^{M_t}\rangle\exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \tag{5.1}$$

Moreover, $(P_t)_{t\ge 0}$ is a Markov transition semigroup on $M(I\!R)$.

Proof. If we replace σ by σ_k, then (5.1) defines the transition semigroup of the SDSM with parameters (a,ρ,σ_k) which exists by Lemma 4.4 and Theorems 2.2 and 4.1. By Lemma 2.1 it is easy to see that

$$\lim_{m\to\infty}\frac{1}{m}\Big(\boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_t,\mu^{M_t}\rangle\exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]\Big)^{1/m} = 0.$$

By Lemma 2.2 and Dawson (1993, p.45), (5.1) really defines a probability measure $P_t(\mu,\cdot)$ on $M(I\!R)$ which is the limit distribution of $X_t^{(k)}$ as $k\to\infty$. To check the Chapman-Kolmogorov equation one simply observes

$$\int_{M(I\!R)}P_r(\mu,d\eta)\int_{M(I\!R)}\langle f,\nu^m\rangle P_t(\eta,d\nu)$$

$$
\begin{aligned}
&= \int_{M(\mathbb{R})} \boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_t, \eta^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] P_r(\mu, d\eta) \\
&= \boldsymbol{Q}^\sigma_{m,f}\Big[\int_{M(\mathbb{R})} \langle Y_t, \eta^{M_t}\rangle P_r(\mu, d\eta) \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
&= \boldsymbol{Q}^\sigma_{m,f}\Big[\boldsymbol{Q}^\sigma_{M_t,Y_t}\Big(\langle Y_r, \mu^{M_r}\rangle \exp\Big\{\frac{1}{2}\int_0^r M_s(M_s-1)ds\Big\}\Big) \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
&= \boldsymbol{Q}^\sigma_{m,f}\Big[\langle Y_{r+t}, \mu^{M_{r+t}}\rangle \exp\Big\{\frac{1}{2}\int_0^{r+t} M_s(M_s-1)ds\Big\}\Big] \\
&= \int_{M(\mathbb{R})} \langle f, \nu^m\rangle P_{r+t}(\eta, d\nu).
\end{aligned}
$$

Therefore, (5.1) defines a Markov transition semigroup (P_t) on $M(\mathbb{R})$. □

Lemma 5.1 *The sequence $\{X_t^{(k)} : t \geq 0\}$ in Theorem 5.1 is a tight in $C([0,\infty), M(\hat{\mathbb{R}}))$.*

Proof. Since $\{\langle 1, X_t^{(k)}\rangle : t \geq 0\}$ is a martingale, one can see as in the proof of Lemma 4.1 that $\{X_t^{(k)} : t \geq 0\}$ satisfies the compact containment condition of Ethier and Kurtz (1986, p.142). Let $\mathcal{L}_k$ denote the generator of $\{X_t^{(k)} : t \geq 0\}$ and let F be given by (4.1) with $f \in C_0^\infty(\mathbb{R}^n)$ and with $\phi_i \in C_\partial^2(\mathbb{R})$. Then

$$
F(X_t^{(k)}) - F(X_0^{(k)}) - \int_0^t \mathcal{L}_k F(X_s^{(k)})ds, \quad t \geq 0,
$$

is a martingale. Since the sequence $\{\sigma_k\}$ is uniformly bounded, the desired tightness follows from Lemma 4.4 and the result of Ethier and Kurtz (1986, p.145). □

Theorem 5.2 *The transition semigroup $(P_t)_{t\geq 0}$ defined by (5.1) has a diffusion realization. Let $\mathcal{D}(\mathcal{L})$ be the union of all functions of the form (4.1) with $f \in C_0^\infty(\mathbb{R}^n)$ and $\phi_i \in C^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C_\partial^2(\mathbb{R}^m)$. If $\{X_t : t \geq 0\}$ is a diffusion process with semigroup $(P_t)_{t\geq 0}$ and initial state $X_0 = \mu \in M(\mathbb{R})$, then it solves the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem.*

Proof. Let $\{X_t^{(k)} : t \geq 0\}$ be as in Theorem 5.1. By passing to a subsequence, we may assume that the distribution $\boldsymbol{P}_k$ of $\{X_t^{(k)} : t \geq 0\}$ converges as $k \to \infty$ to some probability measure $\boldsymbol{P}$ on $C([0,\infty), M(\hat{\mathbb{R}}))$. Let $\phi_n \in C^2(\mathbb{R})^+$ be such that $\phi_n(x) = 0$ when $\|x\| \leq n$ and $\phi_n(x) = 1$ when $\|x\| \geq 2n$ and $\|\phi_n'\| \to 0$ as $n \to \infty$. Fix $u > 0$ and let m_n be such that $\phi_{m_n}(x) \leq 2T_t\phi_n(x)$ for all $0 \leq t \leq u$ and $x \in \mathbb{R}$. For any $\alpha > 0$, the paths $w \in C([0,\infty), M(\hat{\mathbb{R}}))$ satisfying $\sup_{0\leq t\leq u}\langle \phi_{m_n}, w_t\rangle > \alpha$ constitute an open subset of $C([0,\infty), M(\hat{\mathbb{R}}))$. Then, by an equivalent condition for weak convergence and a martingale inequality,

$$\begin{aligned}
& \boldsymbol{P}\Big\{\sup_{0\le t\le u} w_t(\{(\partial)\}) > \alpha\Big\} \\
\le\ & \boldsymbol{P}\Big\{\sup_{0\le t\le u} \langle \phi_{m_n}, w_t\rangle > \alpha\Big\} \\
\le\ & \liminf_{k\to\infty} \boldsymbol{P}_k\Big\{\sup_{0\le t\le u} \langle \phi_{m_n}, w_t\rangle > \alpha\Big\} \\
\le\ & \liminf_{k\to\infty} \boldsymbol{P}_k\Big\{2\sup_{0\le t\le u} \langle \hat{T}_{u-t}\phi_n, w_t\rangle > \alpha\Big\} \\
\le\ & \sup_{k\ge 1} \frac{4}{\alpha^2}\boldsymbol{P}_k\{\langle \hat{T}_{u-t}\phi_n, w_t\rangle^2\} \\
\le\ & \sup_{k\ge 1} \frac{4}{\alpha^2}\boldsymbol{P}_k\{|\langle \hat{T}_{u-t}\phi_n, w_t\rangle - \langle \hat{T}_u\phi_n, \mu_k\rangle|^2\} + \sup_{k\ge 1}\frac{4}{\alpha^2}\langle \hat{T}_u\phi_n, \mu_k\rangle^2.
\end{aligned}$$

As in the proof of Theorem 4.1, one can see that the right hand side goes to zero as $n \to \infty$. Then $\boldsymbol{P}$ is supported by $C([0,\infty), M(I\!R))$ and $\boldsymbol{P}_k \to \boldsymbol{P}$ by the weak convergence of probability measures on $C([0,\infty), M(I\!R))$. Using Theorem 5.1 one can show that $\boldsymbol{P}$ satisfies the Markov property. The strong Markov property holds since $(P_t)_{t\ge 0}$ is Feller. To see the second assertion, one may simply check that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is a restriction of the generator of $(P_t)_{t\ge 0}$. □

6 Rescaled limits

In this section, we study the rescaled limits of the SDSM constructed in the last section. Given any $\theta > 0$, we defined the operator K_θ on $M(I\!R)$ by $K_\theta\mu(B) = \mu(\{\theta x : x \in B\})$. For a function $h \in B(I\!R)$ let $h_\theta(x) = h(\theta x)$.

Lemma 6.1 *Suppose that $\{X_t : t \ge 0\}$ is an SDSM with parameters (a, ρ, σ). Let $X_t^\theta = \theta^{-2}K_\theta X_{\theta^2 t}$. Then $\{X_t^\theta : t \ge 0\}$ is an SDSM with parameters $(a, \rho_\theta, \sigma_\theta)$.*

Proof. We shall compute the generator of $\{X_t^\theta : t \ge 0\}$. Let $F(\mu) = f(\langle\phi, \mu\rangle)$ with $f \in C_0^\infty(I\!R)$ and $\phi \in C^2(I\!R)$. Note that $F \circ K_\theta(\mu) = F(K_\theta\mu) = f(\langle \phi_{1/\theta}, \mu\rangle)$. By the theory of Markov processes, $\{K_\theta X_t : t \ge 0\}$ has generator $\mathcal{L}^\theta$ such that $\mathcal{L}^\theta F = \mathcal{L}(F \circ K_\theta)(K_{1/\theta}\mu)$. Since

$$\frac{d}{dx}\phi_{1/\theta}(x) = \frac{1}{\theta}(\phi')_{1/\theta}(x) \quad \text{and} \quad \frac{d^2}{dx^2}\phi_{1/\theta}(x) = \frac{1}{\theta^2}(\phi'')_{1/\theta}(x),$$

it is easy to check that

$$\mathcal{L}^\theta F(\mu) \;=\; \frac{1}{2\theta^2} f'(\langle\phi, \mu\rangle)\langle a_\theta \phi'', \mu\rangle$$

$$+\frac{1}{2\theta^2}f''(\langle\phi,\mu\rangle)\int_{I\!R^2}\rho_\theta(x-y)\phi'(x)\phi'(y)\mu(dx)\mu(dy)$$
$$+\frac{1}{2}f''(\langle\phi,\mu\rangle)\langle\sigma_\theta\phi^2,\mu\rangle.$$

By a similar calculation one may check that $\{\theta^{-2}K_\theta X_t : t \geq 0\}$ has generator $\mathcal{L}_\theta$ such that

$$\begin{aligned}\mathcal{L}_\theta F(\mu) &= \frac{1}{2\theta^2}f'(\langle\phi,\mu\rangle)\langle a_\theta\phi'',\mu\rangle \\ &\quad+\frac{1}{2\theta^2}f''(\langle\phi,\mu\rangle)\int_{I\!R^2}\rho_\theta(x-y)\phi'(x)\phi'(y)\mu(dx)\mu(dy) \\ &\quad+\frac{1}{2\theta^2}f''(\langle\phi,\mu\rangle)\langle\sigma_\theta\phi^2,\mu\rangle.\end{aligned}$$

Then $\{X_t^\theta : t \geq 0\}$ has the right generator. □

Theorem 6.1 *Suppose that $\{X_t : t \geq 0\}$ is an SDSM with parameters $(a, \rho_\theta, \sigma_\theta)$ with $|c(x)| \geq \epsilon > 0$ for all $x \in I\!R$. Then there is a $\lambda\times\lambda\times\boldsymbol{P}_\mu$-measurable function $X_t(w,x)$ such that $\boldsymbol{P}_\mu\{w \in C([0,\infty), M(I\!R)) : X_t(w,dx)$ is absolutely continuous with respect to the Lebesgue measure with density $X_t(w,x)$ for λ-a.e. $t > 0\} = 1$. Moreover, for $\lambda\times\lambda$-a.e. $(t,x) \in [0,\infty)\times I\!R$ we have*

$$\begin{aligned}\boldsymbol{E}_\mu\{X_t(x)^2\} &= \int_{I\!R^2} p_t^2(y,z;x,x)\mu(dx)\mu(dy) \\ &\quad+2\int_0^t ds\int_{I\!R}\mu(dy)\int_{I\!R}\sigma(z)p_t^2(z,z;x,x)dz. \end{aligned}\tag{6.1}$$

Proof. Let $r_1 > 0$ and $r_2 > 0$. By Theorem 2.1 we have

$$\begin{aligned}&\boldsymbol{E}_\mu\{\langle g^1_{\epsilon r_1}(x,\cdot),X_t\rangle\langle g^1_{\epsilon r_2}(x,\cdot),X_t\rangle\} \\ =\;&\boldsymbol{E}_\mu\{\langle g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot),X_t^2\rangle\} \\ =\;&\langle T_t^2 g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot),\mu^2\rangle \\ &+2\int_0^t\langle T_{t-s}\phi_{12}T_s^2 g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot),\mu^2\rangle ds \\ =\;&\int_{I\!R^2}T_t^2 g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot)(y,z)\mu(dy)\mu(dz) \\ &+2\int_0^t ds\int_{I\!R}\mu(dy)\int_{I\!R}\sigma(z)T_s^2 g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot)(z,z)p_{t-s}(y,z)dz.\end{aligned}$$

Observe that

$$T_t^2 g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot)(y,z) = \int_{I\!R^2} g^1_{\epsilon r_1}(x,z_1)g^1_{\epsilon r_2}(x,z_2)p_t^2(y,z;z_1,z_2)dz_1dz_2,$$

which converges to $p_t^2(y,z;x,x)$ boundedly as $r_1 \to 0$ and $r_2 \to 0$. Note also that

$$\begin{aligned} &\int_{I\!R} \sigma(z) T_s^2 g^1_{\epsilon r_1}(x,\cdot)\otimes g^1_{\epsilon r_2}(x,\cdot)(z,z) p_{t-s}(y,z)dz \\ \leq\ & \text{const}\cdot\|\sigma\|\frac{1}{\sqrt{s}}\int_{I\!R} P_{\epsilon s} g^1_{\epsilon r_1}(x;\cdot)(z) g^1_{\epsilon(t-s)}(y,z)dz \\ \leq\ & \text{const}\cdot\|\sigma\|\frac{1}{\sqrt{s}} g^1_{\epsilon(t+r_1)}(y,x) \\ \leq\ & \text{const}\cdot\|\sigma\|\frac{1}{\sqrt{st}}. \end{aligned}$$

By dominated convergence theorem we get

$$\begin{aligned} &\lim_{r_1,r_2\to 0} \boldsymbol{E}_\mu\{\langle g^1_{\epsilon r_1}(x,\cdot),X_t\rangle\langle g^1_{\epsilon r_2}(x,\cdot),X_t\rangle\} \\ =\ & \int_{I\!R^2} p_t^2(y,z;x,x)\mu(dy)\mu(dz) \\ &\quad +2\int_0^t ds\int_{I\!R}\mu(dy)\int_{I\!R}\sigma(z)p_t^2(z,z;x,x)p_{t-s}(y,z)dz. \end{aligned}$$

Then it is easy to check that

$$\lim_{r_1,r_2\to 0}\int_0^T dt\int_{I\!R}\boldsymbol{E}_\mu\{\langle g^1_{\epsilon r_1}(x,\cdot)-g^1_{\epsilon r_2}(x,\cdot),X_t\rangle^2\}dx = 0$$

for each $T > 0$, so there is a $\lambda\times\lambda\times\boldsymbol{P}_\mu$-measurable function $X_t(w,x)$ such that (6.1) holds and

$$\lim_{r\to 0}\langle g^1_{\epsilon r_1}(x,\cdot),X_t\rangle = X_t(w,x) \quad \text{in} \quad L^2(\lambda\times\lambda\times\boldsymbol{P}_\mu). \tag{6.2}$$

Observe that

$$\begin{aligned} &\int_0^T \boldsymbol{E}_\mu\Big\{\Big|\langle\phi,X_t\rangle-\int_{I\!R}\phi(x)X_t(x)dx\Big|^2\Big\}dt \\ \leq\ & \int_0^T \boldsymbol{E}_\mu\left\{\langle\phi-P_{\epsilon r}\phi,X_t\rangle^2\right\}dt \\ &\quad +\int_0^T \boldsymbol{E}_\mu\Big\{\Big|\langle P_{\epsilon r}\phi,X_t\rangle-\int_{I\!R}\phi(x)X_t(x)dx\Big|^2\Big\}dt, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} &\boldsymbol{E}_\mu\Big\{\Big|\langle P_{\epsilon r}\phi,X_t\rangle-\int_{I\!R}\phi(x)X_t(x)dx\Big|^2\Big\} \\ =\ & \boldsymbol{E}_\mu\Big\{\Big|\int_{I\!R}X_t(dx)\int_{I\!R}\phi(x)g^1_{\epsilon r}(y,x)dx-\int_{I\!R}\phi(x)X_t(x)dx\Big|^2\Big\} \\ =\ & \boldsymbol{E}_\mu\Big\{\Big|\int_{I\!R}[\langle g^1_{\epsilon r}(\cdot,x),X_t\rangle-X_t(x)]\phi(x)dx\Big|^2\Big\} \\ \leq\ & \int_{I\!R}\boldsymbol{E}_\mu\left\{|\langle g^1_{\epsilon r}(\cdot,x),X_t\rangle-X_t(x)|^2\right\}dx\int_{I\!R}\phi(x)^2dx. \end{aligned}$$

By this and (6.2) we get

$$\lim_{r\to 0}\int_0^T \boldsymbol{E}_\mu\Big\{\Big|\langle P_{\epsilon r}\phi, X_t\rangle - \int_{I\!R}\phi(x)X_t(x)dx\Big|^2\Big\}dt = 0.$$

On the other hand,

$$\lim_{r\to 0}\boldsymbol{E}_\mu\left\{\langle \phi - P_{\epsilon r}\phi, X_t\rangle^2\right\} \le \lim_{r\to 0}\boldsymbol{E}_\mu\left\{\langle |\phi - P_{\epsilon r}\phi|^2, X_t\rangle\right\} = 0$$

so letting $r \to 0$ in (6.3) we see that

$$\int_0^T \boldsymbol{E}_\mu\Big\{\Big|\langle \phi, X_t\rangle - \int_{I\!R}\phi(x)X_t(x)dx\Big|^2\Big\}dt = 0,$$

completing the proof. □

By Theorem 6.1, for $\lambda\times\lambda$-a.e. $(t,x)\in[0,\infty)\times I\!R$ we have

$$\begin{aligned}\boldsymbol{E}_\mu\{X_t(x)^2\} &\le \text{const}\cdot\Big[\frac{1}{\sqrt{t}}\langle 1,\mu\rangle\int_{I\!R} g^1_{\epsilon t}(x,y)\mu(dy) \\ &\qquad + \int_0^t \frac{ds}{\sqrt{s}}\int_{I\!R}\mu(dy)\int_{I\!R}\|\sigma\| g^1_{\epsilon s}(z,x) g^1_{\epsilon(t-s)}(z,x)dz\Big] \\ &\le \text{const}\cdot\Big[\frac{1}{\sqrt{t}}\langle 1,\mu\rangle + \sqrt{t}\|\sigma\|\Big]\int_{I\!R} g^1_{\epsilon t}(x,y)\mu(dy). \end{aligned} \tag{6.4}$$

Theorem 6.2 *Suppose that $\{X_t : t\ge 0\}$ is an SDSM with parameters (a,ρ,σ) with $|c(x)|\ge\epsilon>0$ for all $x\in I\!R$. Let $X^\theta_t = \theta^{-2}K_\theta X_{\theta^2 t}$. Assume $a(x)\to a_\partial$, $\sigma(x)\to\sigma_\partial$ and $\rho(x)\to 0$ as $|x|\to\infty$. Then the conditional law of $\{X^\theta_t : t\ge 0\}$ given $X^\theta_0=\mu\in M(I\!R)$ converges to that of a super Brownian motion with underlying generator $(a_\partial/2)\Delta$ and uniform branching density σ_∂.*

Proof. Since $\|\sigma_\theta\| = \|\sigma\|$ and $X^\theta_0=\mu$, as in the proof of Lemma 5.1 one can see that the family $\{X^\theta_t : t\ge 0\}$ is tight in $C([0,\infty), M(I\!R))$. Choose any sequence $\theta_k\to\infty$ such that the law of $\{X^{\theta_k}_t : t\ge 0\}$ converges to some probability measure $\boldsymbol{Q}$ on $C([0,\infty), M(I\!R))$. We need to prove that $\boldsymbol{Q}$ is a solution of the martingale problem for the super Brownian motion. By the representation of Skorokhod, we can construct SDSM's $\{X^{(k)}_t : t\ge 0\}$ and $\{X^{(0)}_t : t\ge 0\}$ such that $\{X^{(k)}_t : t\ge 0\}$ and $\{X^{\theta_k}_t : t\ge 0\}$ have identical laws, $\{X^{(0)}_t : t\ge 0\}$ has the law $\boldsymbol{Q}$ and $\{X^{(k)}_t : t\ge 0\}$ converges a.s. to $\{X^{(0)}_t : t\ge 0\}$ in $C([0,\infty), M(I\!R))$. Let $F(\mu)=f(\langle\phi,\mu\rangle)$ with $f\in C^\infty_0(I\!R)$ and $\phi\in C^2(I\!R)$. Then for each $k\ge 0$,

$$F(X^{(k)}_t) - F(X^{(k)}_0) - \int_0^t \mathcal{L}_k F(X^{(k)}_s)ds, \quad t\ge 0, \tag{6.5}$$

is a martingale, where $\mathcal{L}_k$ is given by

$$\begin{aligned}\mathcal{L}_k F(\mu) &= \frac{1}{2} f'(\langle \phi, \mu \rangle) \langle a_{\theta_k} \phi'', \mu \rangle \\ &\quad + \frac{1}{2} f''(\langle \phi, \mu \rangle) \int_{I\!R^2} \rho_{\theta_k}(x-y) \phi'(x) \phi'(y) \mu(dx) \mu(dy) \\ &\quad + \frac{1}{2} f''(\langle \phi, \mu \rangle) \langle \sigma_{\theta_k} \phi^2, \mu \rangle.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_0^t \boldsymbol{E}\{|f'(\langle \phi, X_s^{(k)} \rangle)| \langle |a_{\theta_k} - a_\partial| \phi'', X_s^{(k)} \rangle\} ds \\ \leq\ & \|f'\| \|\phi''\| \int_0^t \boldsymbol{E}\{\langle |a_{\theta_k} - a_\partial|, X_s^{(k)} \rangle\} ds \\ \leq\ & \|f'\| \|\phi''\| \int_0^t \langle T_s |a_{\theta_k} - a_\partial|, \mu \rangle ds \\ \leq\ & \|f'\| \|\phi''\| \int_0^t ds \int_{I\!R} \mu(dx) \int_{I\!R} |a_{\theta_k}(y) - a_\partial| p_s(x,y) dy,\end{aligned}$$

which goes to zero as $k \to \infty$. In the same way, one sees that

$$\int_0^t \boldsymbol{E}\{|f''(\langle \phi, X_s^{(k)} \rangle)| \langle |\sigma_{\theta_k} - \sigma_\partial| \phi^2, X_s^{(k)} \rangle\} ds$$

also goes to zero as $k \to \infty$. Using the density process of $\{X_t^{(k)} : t \geq 0\}$ we have the following estimates

$$\begin{aligned}&\boldsymbol{E}\Big| f''(\langle \phi, X_s^{(k)} \rangle) \int_{I\!R^2} \rho_{\theta_k}(x-y) \phi'(x) \phi'(y) X_s^{(k)}(dx) X_s^{(k)}(dy) \Big| \\ \leq\ & \|f''\| \int_{I\!R^2} |\rho_{\theta_k}(x-y)| |\phi'(x)\phi'(y)| \boldsymbol{E}\{X_s^{(k)}(x) X_s^{(k)}(y)\} dxdy \\ \leq\ & \|f''\| \int_{I\!R^2} |\rho_{\theta_k}(x-y)| |\phi'(x)\phi'(y)| \boldsymbol{E}\{X_s^{(k)}(x)^2\}^{1/2} \boldsymbol{E}\{X_s^{(k)}(y)^2\}^{1/2} dxdy \\ \leq\ & \|f''\| \Big(\int_{I\!R^2} |\rho_{\theta_k}(x-y)|^2 |\phi'(x)\phi'(y)|^2 dxdy \int_{I\!R^2} \boldsymbol{E}\{X_s^{(k)}(x)^2\} \boldsymbol{E}\{X_s^{(k)}(y)^2\} dxdy \Big)^{1/2} \\ \leq\ & \|f''\| \Big(\int_{I\!R^2} |\rho_{\theta_k}(x-y)|^2 |\phi'(x)\phi'(y)|^2 dxdy \Big)^{1/2} \int_{I\!R} \boldsymbol{E}\{X_s^{(k)}(x)^2\} dx.\end{aligned}$$

By (6.4), for any fixed $t \geq 0$,

$$\int_0^t ds \int_{I\!R} \boldsymbol{E}\{X_s^{(k)}(x)^2\} dx$$

is uniformly bounded in $k \geq 1$. Since $\rho_{\theta_k}(x-y) \to 0$ for $\lambda \times \lambda$-a.e. $(x,y) \in I\!R^2$ and since $\|\rho_{\theta_k}\| = \|\rho\|$, we have

$$\lim_{k \to \infty} \int_{I\!R^2} |\rho_{\theta_k}(x-y)|^2 |\phi'(x)\phi'(y)|^2 dxdy = 0$$

when $\phi' \in L^2(\lambda)$. Then letting $k \to \infty$ in (6.5) we see that $\{X_t^{(0)} : t \geq 0\}$ is a solution of the martingale problem of the super Brownian motion. $\square$

7 Measure-valued catalysts

In this section, we assume $|c(x)| \geq \epsilon > 0$ for all $x \in I\!R$ and give construction for a class of SDSM with measure-valued catalysts. We start from the construction of a class of measure-valued dual processes. Let $M_B(I\!R)$ denote the space of Radon measures ζ on $I\!R$ to which there correspond constants $b(\zeta) > 0$ and $l(\zeta) > 0$ such that

$$\zeta([x, x + l(\zeta)]) \leq b(\zeta) l(\zeta), \qquad x \in I\!R. \tag{7.1}$$

Clearly, $M_B(I\!R)$ contains all finite measures and all Radon measures which are absolutely continuous with respect to the Lebesgue measure with bounded densities. Let $M_B(I\!R^m)$ denote the space of Radon measures ν on $I\!R^m$ such that

$$\nu(dx_1, \cdots, dx_m) = f(x_1, \cdots, x_m) dx_1, \cdots, dx_{m-1} \zeta(dx_m) \tag{7.2}$$

for some $f \in C(I\!R^m)$ and $\zeta \in M_B(I\!R)$. We endow $M_B(I\!R^m)$ with the topology of vague convergence. Let $M_A(I\!R^m)$ denote the subspace of $M_B(I\!R^m)$ comprising of measures which are absolutely continuous with respect to the Lebesgue measure and have bounded densities. For $f \in C(I\!R^m)$, we define $\lambda_f^m \in M_A(I\!R^m)$ by $\lambda_f^m(dx) = f(x)dx$. Let $\boldsymbol{M}_B$ be the topological union of $\{M_B(I\!R^m) : m = 1, 2, \cdots\}$.

Let $\eta \in M_B(I\!R)$ and let Φ_{ij} be the mapping from $M_A(I\!R^m)$ to $M_B(I\!R^{m-1})$ determined by

$$\begin{aligned} & \Phi_{ij}\mu(dx_1, \cdots, dx_{m-1}) \\ = \; & \mu'(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}) dx_1 \cdots dx_{m-2} \eta(dx_{m-1}), \end{aligned} \tag{7.3}$$

where μ' denotes the Radon-Nikodym derivative of μ with respect to the m-dimensional Lebesgue measure, and x_{m-1} is in the places of the ith and the jth variables of μ' on the right hand side.

Lemma 7.1 *If $\zeta \in M_B(I\!R)$ satisfies (7.1), then*

$$\int_{I\!R} p_t(x, y) \zeta(dy) \leq \frac{1}{\sqrt{2\pi\epsilon t}} h(\epsilon, \zeta; t), \qquad t \geq 0, x \in I\!R, \tag{7.4}$$

where

$$h(\epsilon, \zeta; t) = const \cdot b(\zeta) \Big[2l(\zeta) + \sqrt{2\pi\epsilon t} \Big], \qquad t \geq 0. \tag{7.5}$$

Proof. Using (1.9) and (7.1) we have

$$\begin{aligned}
\int_{\mathbb{R}} p_t(x,y)\zeta(dy) &\leq \text{const}\cdot\int_{\mathbb{R}} g_{\epsilon t}(x,y)\zeta(dy)\\
&\leq \text{const}\cdot\frac{2b(\zeta)l(\zeta)}{\sqrt{2\pi\epsilon t}}\sum_{k=0}^{\infty}\exp\Big\{-\frac{k^2 l(\zeta)^2}{2\epsilon t}\Big\}\\
&\leq \text{const}\cdot\frac{b(\zeta)}{\sqrt{2\pi\epsilon t}}\Big[2l(\zeta)+\int_{\mathbb{R}}\exp\Big\{-\frac{y^2}{2\epsilon t}\Big\}dy\Big]\\
&\leq \text{const}\cdot\frac{b(\zeta)}{\sqrt{2\pi\epsilon t}}\Big[2l(\zeta)+\sqrt{2\pi\epsilon t}\Big],
\end{aligned}$$

as desired. □

Lemma 7.2 *Let $\eta\in M_B(\mathbb{R})$. For any integer $n\geq 1$, define $\eta_n\in M_A(\mathbb{R})$ by*

$$\eta_n(dx)=\eta\Big(\Big(\frac{il(\eta)}{n},\frac{(i+1)l(\eta)}{n}\Big]\Big)\frac{ndx}{l(\eta)},\qquad x\in\Big(\frac{il(\eta)}{n},\frac{(i+1)l(\eta)}{n}\Big],\tag{7.6}$$

where $i=\cdots,-2,-1,0,1,2,\cdots$. Then $\eta_n\to\eta$ by weak convergence and

$$\eta_n([x,x+l(\eta)])\leq 2b(\eta)l(\eta),\qquad x\in\mathbb{R}.\tag{7.7}$$

Proof. For any $x\in\mathbb{R}$ there is an integer i such that

$$[x,x+l(\eta)]\subset\Big(\frac{il(\eta)}{n},\frac{(i+1)l(\eta)}{n}+l(\eta)\Big].$$

Therefore, we have

$$\begin{aligned}
\eta_n([x,x+l(\eta)]) &\leq \eta_n\Big(\Big(\frac{il(\eta)}{n},\frac{(i+1)l(\eta)}{n}+l(\eta)\Big]\Big)\\
&= \eta\Big(\Big(\frac{il(\eta)}{n},\frac{(i+1)l(\eta)}{n}+l(\eta)\Big]\Big)\\
&\leq \eta\Big(\Big(\frac{il(\eta)}{n},\frac{il(\eta)}{n}+2l(\eta)\Big]\Big)\\
&\leq 2b(\eta)l(\eta).
\end{aligned}$$

That is, (7.7) holds. □

We may also regards T_t^m as a semigroup of operators on $M_B(\mathbb{R}^m)$ determined by

$$T_t^m\nu(dx)=\int_{\mathbb{R}^m}p_t^m(x,y)\nu(dy)dx,\qquad t>0,x\in\mathbb{R}^m.\tag{7.8}$$

Of course, T_t^m maps $M_B(\mathbb{R}^m)$ to $M_A(\mathbb{R}^m)$, and for $f \in C(\mathbb{R}^m)$ we have

$$T_t^m \lambda_f^m(dx) = T_t^m f(x)dx, \qquad t > 0, x \in \mathbb{R}^m. \tag{7.9}$$

Suppose that $\{M_t : t \geq 0\}$ is a nonnegative integer-valued pure jump Markov process with transition intensities $q_{i,j}$ such that $q_{i,i-1} = i(i-1)/2$ and $q_{i,j} = 0$ for all other pairs (i,j). Let $\tau_0 = 0$ and $\tau_{M_0} = \infty$, and let $\{\tau_k : 1 \leq k \leq M_0 - 1\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$. Let $\{\Gamma_k : 1 \leq k \leq M_0 - 1\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\boldsymbol{P}\{\Gamma_k = \Phi_{i,j} | M(\tau_k^-) = l\} = \frac{1}{l(l-1)}, \quad 1 \leq i \neq j \leq l, \tag{7.10}$$

where $\Phi_{i,j}$ is defined by (7.3). Then

$$Z_t = T_{t-\tau_k}^{M_{\tau_k}} \Gamma_k T_{\tau_k - \tau_{k-1}}^{M_{\tau_{k-1}}} \Gamma_{k-1} \cdots T_{\tau_2-\tau_1}^{M_{\tau_1}} \Gamma_1 T_{\tau_1}^{M_0} Z_0, \quad \tau_k \leq t < \tau_{k+1}, 0 \leq k \leq M_0 - 1, \tag{7.11}$$

defines a Markov process $\{Z_t : t \geq 0\}$ taking values from $\boldsymbol{M}_B$. Of course, $\{(M_t, Z_t) : t \geq 0\}$ is also a Markov process. We shall suppress the dependence of $\{Z_t : t \geq 0\}$ on η and let $\boldsymbol{Q}_{m,\nu}^\eta$ denote the expectation given $M_0 = m$ and $Z_0 = \nu \in M_B(\mathbb{R}^m)$.

Lemma 7.3 *If $\eta \in M_B(\mathbb{R})$ and if $\nu \in M_B(\mathbb{R}^m)$ is given by (7.2), then*

$$\begin{aligned} &\boldsymbol{Q}_{m,\nu}^\eta \Big[\langle Z_t', \mu^{M_t} \rangle \exp \Big\{ \frac{1}{2} \int_0^t M_s(M_s - 1) ds \Big\} \Big] \\ \leq\;& \|f\| h(\epsilon, \zeta; t) \Big[\langle 1, \mu \rangle^m / \sqrt{t} + \sum_{k=1}^{m-1} 2^k m^k (m-1)^k h(\epsilon, \eta; t)^k \langle 1, \mu \rangle^{m-k} t^{k/2} \Big]. \end{aligned} \tag{7.12}$$

(Note that the left hand side of (7.12) is well defined since $Z_t \in M_A(\mathbb{R})$ a.s. for each $t > 0$ by (7.11).)

Proof. For $0 \leq k \leq m-1$ set

$$A_k = \boldsymbol{Q}_{m,\nu}^\eta \Big[\langle Z_t', \mu^{M_t} \rangle \exp \Big\{ \frac{1}{2} \int_0^t M_s(M_s - 1) ds \Big\} 1_{\{\tau_k \leq t < \tau_{k+1}\}} \Big].$$

By (7.2) and Lemma 7.1,

$$A_0 = \langle (T_t^m \nu)', \mu^m \rangle \leq \|f\| h(\epsilon, \zeta; t) \langle 1, \mu \rangle^m / \sqrt{t}.$$

By the construction (7.11), for $1 \leq k \leq m-1$, A_k is equal to

$$\frac{m!(m-1)!}{2^k (m-k)!(m-k-1)!} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t \boldsymbol{E} \langle (T_{t-s_k}^{m-k} \Gamma_k \cdots T_{s_2-s_1}^{m-1} \Gamma_1 T_{s_1}^m \nu)', \mu^{m-k} \rangle ds_k.$$

Observe that

$$\int_{s_{k-1}}^t \frac{ds_k}{\sqrt{t-s_k}\sqrt{s_k-s_{k-1}}} \le \frac{2\sqrt{2}}{\sqrt{t-s_{k-1}}} \int_{(t+s_{k-1})/2}^t \frac{ds_k}{\sqrt{t-s_k}} \le \frac{4\sqrt{t}}{\sqrt{t-s_{k-1}}}. \tag{7.13}$$

By (7.8) we have $T_s^{m-k}\lambda_h^{m-k} \le \lambda_{\|h\|}^{m-k}$ for $h \in C(I\!R^{m-k})$. Then using (7.13) and Lemma 7.1 inductively we get

$$\begin{aligned} A_k &\le \frac{m!(m-1)!\|f\|}{2^k(m-k)!(m-k-1)!} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t \frac{h(\epsilon,\zeta;t)h(\epsilon,\eta;t)^k\langle 1,\mu\rangle^{m-k}}{\sqrt{t-s_k}\cdots\sqrt{s_2-s_1}\sqrt{s_1}} ds_k \\ &\le \frac{2^k m!(m-1)!\|f\|}{(m-k)!(m-k-1)!} h(\epsilon,\zeta;t)h(\epsilon,\eta;t)^k\langle 1,\mu\rangle^{m-k}t^{k/2} \\ &\le 2^k m^k(m-1)^k\|f\|h(\epsilon,\zeta;t)h(\epsilon,\eta;t)^k\langle 1,\mu\rangle^{m-k}t^{k/2}. \end{aligned}$$

Then we have the conclusion. □

Lemma 7.4 *Suppose $\eta \in M_B(I\!R)$ and define $\eta_n \in M_A(I\!R)$ as in Lemma 7.2. Assume that $\mu_n \to \mu$ weakly as $n \to \infty$. Then we have*

$$\begin{aligned} & \boldsymbol{Q}^{\eta}_{m,\nu}\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = & \lim_{n\to\infty} \boldsymbol{Q}^{\eta_n}_{m,\nu}\Big[\langle Z_t', \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \end{aligned} \tag{7.14}$$

Proof. By the construction (7.11) we have

$$\begin{aligned} & \boldsymbol{Q}^{\eta_n}_{m,\nu}\Big[\langle Z_t', \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = & \langle (T_t^m\nu)', \mu_n^m\rangle \\ + & \frac{1}{2}\sum_{i,j=1,i\neq j}^m \int_0^t \boldsymbol{Q}^{\eta_n}_{m-1,\Phi_{ij}T_u^m\nu}\Big[\langle Z_{t-u}', \mu_n^{M_{t-u}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\}\Big]du. \end{aligned} \tag{7.15}$$

For any $h \in C(I\!R^2)$,

$$\begin{aligned} & \boldsymbol{Q}^{\eta_n}_{1,\Phi_{12}\lambda_h^2}\Big[\langle Z_t', \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = & \boldsymbol{Q}^{\eta_n}_{1,\Phi_{21}\lambda_h^2}\Big[\langle Z_t', \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = & \int_{I\!R^2} h(y,y)p_t(x,y)\mu_n(dx)\eta_n(dy). \end{aligned} \tag{7.16}$$

If $f, g \in C(I\!R)^+$ have bounded supports, then we have $f(x)\mu_n(dx) \to f(x)\mu(dx)$ and $g(y)\eta_n(dy) \to g(y)\eta(dy)$ by weak convergence, so that

$$\lim_{n\to\infty}\int_{I\!R^2} f(x)g(y)p_t(x,y)\mu_n(dx)\eta_n(dy) = \int_{I\!R^2} f(x)g(y)p_t(x,y)\mu(dx)\eta(dy).$$

Since $\{\mu_n\}$ is tight and η_n satisfies (7.7), one may see that $\{p_t(x,y)\mu_n(dx)\eta_n(dy)\}$ is a tight sequence and hence $p_t(x,y)\mu_n(dx)\eta_n(dy) \to p_t(x,y)\mu(dx)\eta(dy)$ by weak convergence. Therefore, the value of (7.16) converges as $n \to \infty$ to

$$\begin{aligned} & \boldsymbol{Q}^{\eta}_{1,\Phi_{12}\lambda_h^2}\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = \ & \boldsymbol{Q}^{\eta}_{1,\Phi_{21}\lambda_h^2}\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = \ & \int_{I\!R^2} h(y,y)p_t(x,y)\mu(dx)\eta(dy). \end{aligned}$$

Applying Lemma 7.3 and bounded convergence we get inductively from (7.15) that

$$\begin{aligned} & \boldsymbol{Q}^{\eta}_{m-1,\Phi_{ij}T_t^m\nu}\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = \ & \lim_{n\to\infty} \boldsymbol{Q}^{\eta_n}_{m-1,\Phi_{ij}T_t^m\nu}\Big[\langle Z_t', \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \end{aligned}$$

for $1 \leq i \neq j \leq m$. Then the result follows from (7.15). □

Let $\eta \in M_B(I\!R)$ and let η_n be defined as in Lemma 7.2. Let σ_n denote the density of η_n with respect to the Lebesgue measure and let $\{X_t^{(n)} : t \geq 0\}$ be an SDSM with parameters (a, ρ, σ_n) and initial state $\mu_n \in M(I\!R)$.

Theorem 7.1 *Assume that $\mu_n \to \mu$ weakly as $n \to \infty$. Then the distribution of $X_t^{(n)}$ converges to a probability measure $P_t(\mu, \cdot)$ on $M(I\!R)$ determined by*

$$\int_{M(I\!R)} \langle f, \nu^m\rangle P_t(\mu, d\nu) = \boldsymbol{Q}^{\eta}_{m,\lambda_f^m}\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \qquad (7.17)$$

Moreover, $(P_t)_{t\geq 0}$ form a Markov transition semigroup on $M(I\!R)$.

Proof. Clearly, if we replace η by η_n, then (7.17) defines the transition semigroup of the SDSM with parameters (a, ρ, σ_n). By Lemma 7.3 it is easy to see that

$$\lim_{m\to\infty} \frac{1}{m}\Big(\boldsymbol{Q}^{\eta}_{m,\lambda_f^m}\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]\Big)^{1/m} = 0.$$

By Lemma 7.4 and Dawson (1993, p.45), (7.17) really defines a probability measure $P_t(\mu, \cdot)$ on $M(I\!R)$ which is the limit distribution of $X_t^{(n)}$ as $n \to \infty$. The Chapman-Kolmogorov equation can be checked as in the proof of Theorem 5.1. □

References

[1] Chow, P.L., *Function space differential equations associated with a stochastic partial differential equation*, Indiana Univ. Math J. **25** (1976), 609-627.

[2] Dawson, D.A., *Measure-valued Markov processes*, In: Lect. Notes. Math. **1541**, 1-260, Springer-Verlag, Berlin (1993).

[3] Dawson, D.A. and K. Fleischmann, K., *Critical branching in a highly fluctuating random medium*, Probab. Th. Rel. Fields **90** (1991), 241-274.

[4] Dawson, D.A. and Fleischmann, K., *Diffusion and reaction caused by a point catalysts*, SIAM J. Appl. Math. **52** (1992), 163-180.

[5] Dawson, D.A. and Vaillancourt, J., *Stochastic McKean-Vlasov equations*, Nonlinear Diff. Eq. Appl. **2**, 199-229.

[6] Dawson, D.A., Vaillancourt, J. and Wang, H., *Stochastic partial differential equations for a class of measure-valued branching diffusions in a random medium*, Ann. Inst. Henri Poincare, Probabilités and Statistiques **36** (2000), 167-180.

[7] Dawson, D.A., Vaillancourt, J. and Wang, H., *Local time for a class of interacting measure-valued diffusions*, to appear (2000).

[8] Durrett, R. and Perkins, E.A., *Rescaled contact processes converge to super-Brownian motion in two or more dimensions*, Probab. Theory and Related Fields, **114** (1999), 309-399.

[9] Evans, S.N. and Pitman, J, *Construction of Markovian coalescents*, Ann. Inst. H. Poincaré Probab. Statist. **34** (1998), 339-383.

[10] Ethier, S.N. and Kurtz, T.G., *Markov Processes: Characterization and Convergence*, Wiley, New York (1986).

[11] Friedman, A., *Partial Differential Equations of Parabolic Type*, Englewood Cliffs, NJ, Prentice Hall (1964).

[12] Hara, T. and Slade, G., *The scaling limit of the incipient infinite cluster in high-dimensional percolation I: Critical exponents*, J. Stat. Phys. **99** (2000), 1075-1168.

[13] Hara, T. and Slade, G., *The scaling limit of the incipient infinite cluster in high-dimensional percolation I: Integrated super-Brownian excursion*, J. Stat. Phys. **41** (2000), 1244-1293.

[14] Kotelenez, P., *Existence, uniqueness and smoothness for a class of function valued stochastic partial differential equations*, Stochastics **41** (1992), 177-199.

[15] Kotelenez, P., *A class of quasilinear stochastic partial differential equations of McKean-Vlasov type with mass conservation*, Probab. Th. Rel. Fields **102** (1995), 159-188.

[16] Walsh, J.B., *An Introduction to Stochastic Partial Differential Equations*, Lect. Notes Math. **1180**, 265-439, Springer-Verlag (1986).

[17] Wang, H., *State classification for a class of measure-valued branching diffusions in a Brownian medium*, Probab. Th. Rel. Fields **109** (1997), 39-55.

[18] Wang, H., *A class of measure-valued branching diffusions in a random medium*, Stochastic Analysis and Applications **16** (1998), 753-786.