

TRAJECTORIAL FLUCTUATIONS OF COX SYSTEMS OF INDEPENDENT MOTIONS

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Abstract

We obtain high-density fluctuation limits for the trajectories of the motions in Cox systems of independent motions in \mathbb{R}^d . The motions are quite general; they include a large class of diffusions, Brownian bridges and fractional Brownian motions. The limits take values in a space of distributions on Wiener space and in general are non-Gaussian.

Key words: Infinite particle system, Cox measure, trajectorial fluctuation, distribution on Wiener space, non-Gaussian random distribution, time-localization.

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1. Introduction

We consider infinite particle systems in \mathbb{R}^d . The usual object of study is the point measure-valued process $N = \{N(t), t \in [0, 1]\}$, where $N(t)$ is the empirical measure of the system at time t , i.e., $N(t) = \sum_j \delta_{\xi_j(t)}$, where $\{\xi_j(t)\}_j$ are the positions of the particles present at that time. For many particle systems it is possible to show that a high-density fluctuation limit of N exists, which is a process with values in $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions on \mathbb{R}^d . We refer to this as the “temporal” approach due to the role of time as a parameter. In many cases a more informative analysis is based on the trajectories of the particles. The system is described by the trajectorial empirical measure $\mathcal{N} = \sum_j \delta_{\xi_j}$, where $\{\xi_j\}_j$ are the paths of the particles. Assuming that the particle motion is continuous, \mathcal{N} is a random point measure on $C \equiv C([0, 1], \mathbb{R}^d)$. The problem now consists in obtaining a high-density fluctuation limit for \mathcal{N} . We call this the “trajectorial” approach. This approach has been taken by Martin-Löf [24], Gorostiza [13], Tanaka and Hitsuda [29], Tanaka [28], Sznitmann [27], Grigorescu [17] and others, for different particle models, including some with branchings or interactions.

By analogy with the temporal approach, one expects that the trajectorial fluctuation limits will take values in a space of distributions on C . In most of the above mentioned papers the question of defining a general state space for the limits was not considered, so the limits were not characterized as random elements of some topological space, or ad hoc spaces were used in some cases. A convenient general nuclear space of distributions on C , denoted here by $\mathcal{S}(C)'$, was constructed by Gorostiza and Nualart [14], and examples were given for the fluctuation limits of the particle systems studied in some of the papers cited above. The nuclear property of $\mathcal{S}(C)'$ is important in connection with the use of the Lévy continuity theorem [25].

For many models the particle system is assumed to start off (at time 0) from a configuration given by a Poisson random measure on \mathbb{R}^d with intensity measure μ , and the fluctuation limit turns out to be Gaussian. If the initial configuration is given instead by a Cox random measure (i.e., a doubly stochastic Poisson measure, which is obtained by randomizing the intensity measure μ of a Poisson measure), we then have a Cox system of random motions. In this case the fluctuation limit is in general non-Gaussian and its characteristic functional is given in terms of the Laplace functional of μ . A simple Cox system of independent Brownian motions was studied by Feldman and Iyer [12] in the temporal approach. High-density temporal fluctuation limits of systems of independent motions are called “density process” (see examples [1, 2, 4, 6, 12, 24, 30]).

In this paper we give a general trajectorial fluctuation limit theorem for Cox systems of independent motions (Theorem 2.1). The particle motions are quite general, since Markov or martingale properties are not needed in this approach (Corollary 2.1); in particular they include fractional Brownian motion, which is non-Markovian. Stochastic models based on Cox measures have many areas of application: physics, biology, ecology, risk theory (see, e.g., [16]).

In some cases it is possible to derive temporal fluctuation limits from trajectorial fluctuation limits by means of a “time-localization” procedure that has been developed by Bojdecki et al [8], and Bojdecki and Gorostiza [5, 7]. Roughly speaking, the time-localization consists in “projecting” a random element of $\mathcal{S}(C)'$ at each time $t \in [0, 1]$ in order to obtain a process with paths in $C([0, 1], \mathcal{S}'(\mathbb{R}^d))$, in a consistent way with the temporal approach. As an example we will show that the results of Feldman and Iyer [12], and even generalizations of them in several

ways, are easily obtained with this method (Theorem 3.1 and Remark 3.3). In another example we consider a Cox system of independent motions which is driven by a (α, d, β) -superprocess. The purpose of this example is to study the fluctuations of a Cox system of motions with a more complex structure than the one considered in [12], and such that the intensity of the Cox measure arises from a specific particle model. For this example, and for trajectorial fluctuations of Cox systems in general, the time-localization of trajectorial results poses new technical difficulties which remain to be solved.

In the Appendix we give some background on the spaces of test functions $\mathcal{S}(C)$ and distributions $\mathcal{S}(C)'$, the space of p -tempered measures $\mathcal{M}_p(\mathbb{R}^d)$, and convergence in distribution of random p -tempered measures via Laplace functionals.

2. Model and results

Fix $p > 0$. Let μ_n be a random measure in $\mathcal{M}_p(\mathbb{R}^d)$, $n = 1, 2, \dots$. Given μ_n , let Π_n be a Poisson random measure on \mathbb{R}^d with intensity measure μ_n . The random point measure on \mathbb{R}^d so obtained is called a Cox measure (see [21], Proposition 10.5, for existence). For each $x \in \mathbb{R}^d$, let $\xi^x = \{\xi^x(t), t \in [0, 1]\}$ denote a continuous process starting from x . For any n we define the Cox particle system by letting the process ξ^x evolve from each point x of Π_n , these processes being independent.

We denote by M^x the distribution of ξ^x on the space $C \equiv C([0, 1], \mathbb{R}^d)$, and (under a suitable measurability assumption) we put

$$d\mathbb{M}_n = dM^x \mu_n(dx), \quad n = 1, 2, \dots \quad (2.1)$$

Thus \mathbb{M}_n is a random measure (σ -finite) on C .

Let \mathcal{N}_n denote the trajectorial empirical measure of the system, i.e.,

$$\mathcal{N}_n = \sum_{j=1}^{\infty} \delta_{\xi^{x_j}}, \quad (2.2)$$

where $\{x_j\}_j$ are the points of Π_n and δ_ω is the Dirac measure at $\omega \in C$. We are interested in the limit behaviour of the trajectorial fluctuation \mathcal{X}_n defined by

$$\mathcal{X}_n = \frac{1}{\sqrt{n}}(\mathcal{N}_n - \mathbb{M}_n), \quad (2.3)$$

when the density of the system increases towards infinity (as $n \rightarrow \infty$). We regard \mathcal{N}_n and \mathcal{X}_n as random elements of $\mathcal{S}(C)'$.

In what follows, \Rightarrow denotes convergence in distribution of random elements of a topological space, $\|\cdot\|_\infty$ is the usual sup norm on C , and $\langle \cdot, \cdot \rangle$ means duality (in particular, integration).

Our main result is the following theorem.

Theorem 2.1. *Assume that*

(i)

$$E\|\xi^x - x\|_\infty^n \leq J^n \sqrt{n!} \quad (2.4)$$

for all $x \in \mathbb{R}^d$, $n = 1, 2, \dots$, and some constant J ;
(ii) the function $x \mapsto E|F(\xi^x)|$ is continuous for each $F \in \mathcal{S}(C)$;
(iii)

$$\frac{\mu_n}{n} \Rightarrow \mu \text{ in } \mathcal{M}_p(\mathbb{R}^d) \text{ as } n \rightarrow \infty, \quad (2.5)$$

for some random element μ of $\mathcal{M}_p(\mathbb{R}^d)$.

Define \mathbb{M} by

$$d\mathbb{M} = dM^x \mu(dx). \quad (2.6)$$

Then $\mathbb{X}_n \Rightarrow \mathbb{X}$ in $\mathcal{S}(C)'$, where \mathbb{X} is a random element of $\mathcal{S}(C)'$ with characteristic functional

$$E \exp\{i\langle \mathbb{X}, F \rangle\} = E \exp\left\{-\frac{1}{2}\langle \mathbb{M}, F^2 \rangle\right\}, \quad F \in \mathcal{S}(C). \quad (2.7)$$

Note that the right-hand side of (2.7) can be written as

$$L_\mu\left(\frac{1}{2}\langle M, F^2 \rangle\right),$$

where L_μ is the Laplace functional of μ .

Observe that if μ is non-random, then \mathbb{X} is Gaussian. In the general case the law of \mathbb{X} is a mixture of Gaussian distributions. By analogy with doubly stochastic Poisson systems, \mathbb{X} can be considered as a “doubly stochastic white noise” on C .

This theorem should be compared to Theorem 1.2 of [7]. The main feature of the present version is that the μ_n are random, satisfying assumption (iii), whereas in that paper we had $\mu_n = n\mu$ with a non-random μ . On the other hand, due to the randomness of μ_n the assumptions on ξ^x have to be slightly stronger than those in [7], but they are sufficiently general to cover interesting examples.

Proof of Theorem 2.1.

Step 1. The measures \mathbb{M}_n and \mathbb{M} can be regarded as (random) elements of $\mathcal{S}(C)'$. This follows from Theorem 2.5 of [8] by assumption (ii), and because the measures μ_n and μ are assumed to be p -tempered (in the terminology of [8], \mathbb{M}_n and \mathbb{M} are *admissible* measures, almost surely). In consequence, \mathbb{X}_n is actually a random element of $\mathcal{S}(C)'$.

Step 2. We prove that for each $F \in \mathcal{S}(C)$ the function $x \mapsto E|F(\xi^x)|$ belongs to the space $C_p(\mathbb{R}^d)$ (see Appendix, A.2). Due to assumption (ii), it suffices to show that the function $x \mapsto (1 + |x|^2)^p E|F(\xi^x)|$ is bounded for each $p > 0$. To simplify the notation we consider the case $d = 1$.

Let $F(x + \omega) = \sum_{n=0}^{\infty} I_n(f(x))(\omega)$, $x \in \mathbb{R}$, $\omega \in C_0 \equiv C_0([0, 1], \mathbb{R})$, be the chaos expansion of $F(x + \cdot)$ in the Wiener space C_0 . By (2.6) of [14] we have

$$|F(\xi^x)| = |F(x + (\xi^x - x))| \leq \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{n!}{(n-k)! \sqrt{k!}} \|\xi^x - x\|_{\infty}^{n-k} 2^{-(j-1)n/2} \|f_n(x)\|_{H_j^{\odot n}}$$

for each $j \geq 1$ (see Appendix, A.1, for the definitions of $H_j^{\odot n}$ and \mathcal{H}_j below). Hence by (2.4) we obtain

$$\begin{aligned}
E|F(\xi^x)| &\leq \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{n!}{(n-k)!\sqrt{k!}} J^{n-k} \sqrt{(n-k)!} 2^{-(j-1)n/2} \|f_n(x)\|_{H_j^{\odot n}} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \sqrt{\binom{n}{k}} J^{n-k} \sqrt{n!} 2^{-(j-1)n/2} \|f_n(x)\|_{H_j^{\odot n}} \\
&\leq \sum_{n=0}^{\infty} (1+J)^n 2^{-(j-1)n/2} \sqrt{n!} \|f_n(x)\|_{H_j^{\odot n}} \\
&\leq \left(\sum_{n=0}^{\infty} (1+J)^{2n} 2^{-(j-1)n} \right)^{1/2} \left(\sum_{n=0}^{\infty} n! \|f_n(x)\|_{H_j^{\odot n}}^2 \right)^{1/2} \\
&= (1 - (1+J)^2 2^{-(j-1)})^{-1/2} \|F(x + \cdot)\|_{\mathcal{H}_j},
\end{aligned}$$

provided that j is sufficiently large (such that $(1+J)^2 2^{-(j-1)} < 1$). Now it suffices to observe that from the definition of $\mathcal{S}(C) = \mathcal{S}(\mathbb{R}^d, \mathcal{H})$ (see the Appendix and [14]) it follows that the function $x \mapsto (1 + |x|^2)^p \|F(x + \cdot)\|_{\mathcal{H}_j}$ is bounded for each $p \geq 0$.

Step 3. As the space $\mathcal{S}(C)$ is Fréchet nuclear, the Lévy continuity theorem and the Bochner-Minlos theorem hold in $\mathcal{S}(C)'$ [25, 18]. Hence, to prove the theorem it suffices to show that

$$E \exp\{i\langle \mathcal{X}_n, F \rangle\} \rightarrow E \exp\left\{-\frac{1}{2}\langle \mathbb{M}, F^2 \rangle\right\} \quad \text{as } n \rightarrow \infty \quad (2.8)$$

for each $F \in \mathcal{S}(C)$. (Note that the function $F \mapsto E \exp\{-\frac{1}{2}\langle \mathbb{M}, F^2 \rangle\}$ is continuous by Theorem 2.5 (b) of [8].)

Generalizing slightly formula (4.3) of [7] we obtain

$$E[\exp\{i\langle \mathcal{N}_n, F \rangle\} | \mu_n] = \exp\{\langle \mu_n, Ee^{iF(\xi)} - 1 \rangle\}, \quad (2.9)$$

hence, by (2.3) and observing that by (2.1), $\langle \mathbb{M}_n, F \rangle = \langle \mu_n, EF(\xi) \rangle$, we have

$$\begin{aligned}
&E[\exp\{i\langle \mathcal{X}_n, F \rangle\} | \mu_n] \\
&= \exp\left\{-\frac{i}{\sqrt{n}} \langle \mu_n, EF(\xi) \rangle\right\} \exp\{\langle \mu_n, Ee^{\frac{i}{\sqrt{n}}F(\xi)} - 1 \rangle\} \\
&= \exp\left\{-\frac{1}{2} \left\langle \frac{\mu_n}{n}, EF^2(\xi) \right\rangle\right\} \exp\{\langle \mu_n, EG_n \rangle\},
\end{aligned} \quad (2.10)$$

where

$$G_n(x) = \exp\left\{\frac{i}{\sqrt{n}} F(\xi^x)\right\} - 1 - \frac{i}{\sqrt{n}} F(\xi^x) + \frac{1}{2n} F^2(\xi^x).$$

As

$$|G_n(x)| \leq \frac{1}{6} \frac{|F^3(\xi^x)|}{n\sqrt{n}}, \quad (2.11)$$

we can write

$$\exp\{\langle \mu_n, EG_n \rangle\} = \exp\left\{\left\langle \frac{\mu_n}{n}, \frac{1}{\sqrt{n}} E(F^3(\xi)H_n) \right\rangle\right\},$$

where $|H_n| \leq \frac{1}{6}$.

Let $f(x) = E|F^3(\xi^x)|$. Since $F \in \mathcal{S}(C)$ implies that $F^3 \in \mathcal{S}(C)$ (Appendix, A.1), by Step 2 we see that $f \in C_p(\mathbb{R}^d)$. The real function $\nu \mapsto \langle \nu, f \rangle$ on $\mathcal{M}_p(\mathbb{R}^d)$ is continuous, hence

$$\left| \left\langle \nu, \frac{1}{\sqrt{n}} E(F^3(\xi) H_n) \right\rangle \right| \leq \frac{1}{6\sqrt{n}} \langle \nu, f \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in ν in a compact subset of $\mathcal{M}_p(\mathbb{R}^d)$. In consequence,

$$\exp\left\{-\frac{1}{2}\langle \nu, EF^2(\xi) \rangle\right\} \exp\left\{\left\langle \nu, \frac{1}{\sqrt{n}} E(F^3(\xi) H_n) \right\rangle\right\} \rightarrow \exp\left\{-\frac{1}{2}\langle \nu, EF^2(\xi) \rangle\right\} \quad \text{as } n \rightarrow \infty$$

uniformly on compact subsets of $\mathcal{M}_p(\mathbb{R}^d)$. On the other hand, again by Step 2, the function

$$\nu \mapsto \exp\left\{-\frac{1}{2}\langle \nu, EF^2(\xi) \rangle\right\}$$

is continuous on $\mathcal{M}_p(\mathbb{R}^d)$. Hence by (2.10) and (2.5) and by Theorem 5.5 of [3], we obtain

$$E[\exp\{i\langle \mathcal{X}_n, F \rangle\} | \mu_n] \Rightarrow \exp\left\{-\frac{1}{2}\langle \mu, EF^2(\xi) \rangle\right\} = \exp\left\{-\frac{1}{2}\langle \mathbb{M}, F^2 \rangle\right\}$$

in the sense of the convergence in distribution of real random variables.

Finally, it suffices to observe that the random variables on the left-hand side of (2.9) are obviously uniformly integrable, so (2.8) follows. \square

Remark 2.1. Suppose that each particle in the system has a “charge”, either $+1$ or -1 , assigned to it with equal probabilities and independently of the charges of the other particles. Then, instead of fluctuations \mathcal{X}_n of the empirical measures one can investigate the limit behaviour of the (trajectorial) “total charge”, i.e.,

$$\mathcal{Y}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{\infty} \varepsilon_j \delta_{\xi^{x_j}},$$

where $\varepsilon_1, \varepsilon_2, \dots$ are Rademacher random variables, independent of the other random quantities involved in the model. Models of this kind have been considered, e.g., in [1, 12] (in the temporal context). It is not difficult to see that in this case, and under the assumptions of Theorem 2.1, we obtain the same result, i.e.,

$$\mathcal{Y}_n \Rightarrow \mathcal{X}$$

in $\mathcal{S}(C)'$, where \mathcal{X} is determined by (2.7). Indeed, an argument similar to the one used to derive formula (2.9) yields

$$E[\exp\{i\langle \mathcal{Y}_n, F \rangle\} | \mu_n] = \exp\left\{\left\langle \mu_n, \frac{1}{2} (Ee^{\frac{i}{\sqrt{n}}F(\xi)} + Ee^{-\frac{i}{\sqrt{n}}F(\xi)}) - 1 \right\rangle\right\},$$

and this again has the form $\exp\{-\frac{1}{2}\langle \frac{\mu_n}{n}, EF^2(\xi) \rangle\} \exp\{\langle \mu_n, EG'_n \rangle\}$, with G'_n satisfying (2.11).

Let us discuss briefly assumption (ii) of the theorem. We have the following simple proposition.

Proposition 2.1. *Under assumption (i) of Theorem 2.1, if the function $x \mapsto \xi^x \in C$ on \mathbb{R}^d is continuous a.s., then assumption (ii) of Theorem 2.1 is satisfied .*

Proof. Fix $F \in \mathcal{S}(C)$. It is known that F is continuous on C [14], hence the function $x \mapsto |F(\xi^x)|$ is continuous. Step 2 of the proof of Theorem 2.1 implies that $\sup_x EF^2(\xi^x) < \infty$, since $F^2 \in \mathcal{S}(C)$. Therefore the random variables $\{F(\xi^x), x \in \mathbb{R}^d\}$ are uniformly integrable, so the claimed property follows. \square

Finally, we give some examples of random motions ξ^x to which Theorem 2.1 applies.

Corollary 2.1. *Assume (2.5). Then $\mathcal{X}_n \Rightarrow \mathcal{X}$ and (2.7) holds for any of the following cases:*

(a) $\xi^x = x + \xi^0$, where

(i) ξ^0 is the standard Brownian motion in \mathbb{R}^d , or

(ii) ξ^0 is a diffusion in \mathbb{R}^d starting at the origin, i.e., a solution of the stochastic differential equation

$$d\xi^0(t) = b(t, \xi^0(t))dt + \sigma(t, \xi^0(t))dB(t), \quad (2.12)$$

$\xi^0(0) = 0$, where B is a standard Brownian motion in \mathbb{R}^d , σ is bounded and b has at most linear growth in the space variable, or

(iii) ξ^0 is the standard Brownian bridge in \mathbb{R}^d , or

(iv) ξ^0 is a fractional Brownian motion starting at the origin with arbitrary Hurst parameter $h \in (0, 1)$.

(b) $\{\xi^x, x \in \mathbb{R}^d\}$ is a system of diffusions in \mathbb{R}^d , i.e., solutions of (2.12) with $\xi^x(0) = x$, where b and σ are Lipschitz and bounded .

Proof. It suffices to apply estimations derived in [7,8], and use Theorem 2.1, Proposition 2.1, and basic properties of stochastic flows [23]. \square

3. Time-localization of a simple Cox system

In this section we consider a particular case of the Cox system discussed in Section 2. Namely, we assume that

$$\mu_n = \rho_n \nu, \quad n = 1, 2, \dots, \quad (3.1)$$

where ν is a fixed non-random measure in $\mathcal{M}_p(\mathbb{R}^d)$, and ρ_1, ρ_2, \dots are real positive random variables such that

$$\frac{\rho_n}{n} \Rightarrow \rho \quad \text{as } n \rightarrow \infty, \quad (3.2)$$

for some non-negative random variable ρ . (2.5) is then satisfied with $\mu = \rho\nu$.

Define

$$d\mathbb{N} = dM^x\nu(dx); \quad (3.3)$$

then, clearly, \mathbb{M}_n and \mathbb{M} defined by (2.1) and (2.6) respectively, have the forms $\mathbb{M}_n = \rho_n\mathbb{N}$ and $\mathbb{M} = \rho\mathbb{N}$. Let \mathcal{X}_n be defined by (2.3). By Theorem 2.1 we obtain immediately the following proposition.

Proposition 3.1. *Under the assumptions of Theorem 2.1,*

$$\mathcal{X}_n \Rightarrow \sqrt{\rho}\mathcal{X}^0 \text{ in } \mathcal{S}(C)', \quad (3.4)$$

where \mathcal{X}^0 is a centered Gaussian random element of $\mathcal{S}(C)'$, independent of ρ , with variance functional

$$E\langle \mathcal{X}_0, F \rangle^2 = \langle \mathbb{N}, F^2 \rangle, \quad F \in \mathcal{S}(C). \quad (3.5)$$

Proof. It suffices to observe that the characteristic function of $\sqrt{\rho}\mathcal{X}^0$ is given by (2.7). \square

Remark 3.1. A careful analysis of the proof of Theorem 2.1 shows that by the simple form of the measure μ_n given by (3.1), the assumptions of Theorem 2.1 can be weakened in this case. Instead of (i) and (ii) it suffices to assume that \mathbb{N} is an admissible measure in the sense of [8].

It turns out that in the present case it is easy to deduce a temporal result (see the Introduction) from Proposition 3.1. For any $r > 0$, consider a Poisson measure in \mathbb{R}^d with intensity measure $r\nu$. Let \mathcal{N}_r^0 denote the corresponding trajectorial empirical measure defined by (2.2), and let

$$N_r^0(t) = \sum_j \delta_{\xi^{x_j}(t)}, \quad t \in [0, 1], \quad (3.6)$$

be the empirical process. Let \mathcal{X}_r^0 and $X_r^0(t)$ denote the trajectorial and temporal fluctuations, respectively, i.e.,

$$\mathcal{X}_r^0 = \frac{1}{\sqrt{r}}(\mathcal{N}_r^0 - r\mathbb{N}) \quad \text{and} \quad X_r^0(t) = \frac{1}{\sqrt{r}}(N_r^0(t) - r\nu).$$

X_r^0 is regarded as a process with paths in $C([0, 1], \mathcal{S}'(\mathbb{R}^d))$.

The following lemma follows immediately from the results of [7].

Lemma 3.1. *Let the processes ξ^x be of one of the types in Corollary 2.1. Then*

- (a) $\mathcal{X}_r^0 \Rightarrow \mathcal{X}^0$ in $\mathcal{S}(C)'$ as $r \rightarrow \infty$, where \mathcal{X}^0 is defined in Proposition 3.1;
- (b) $X_r^0 \Rightarrow X^0$ in $C([0, 1], \mathcal{S}'(\mathbb{R}^d))$ as $r \rightarrow \infty$, where X^0 is a continuous centered Gaussian process in $\mathcal{S}'(\mathbb{R}^d)$ with covariance functional

$$E\langle X^0(s), \varphi \rangle \langle X^0(t), \psi \rangle = \int_{\mathbb{R}^d} E\varphi(\xi^x(s))\psi(\xi^x(t))\nu(dx), \quad s, t \in [0, 1], \varphi, \psi \in \mathcal{S}(\mathbb{R}^d).$$

Remark 3.2.

- (a) Part (a) of the lemma follows also, of course, from Proposition 3.1.

- (b) Here again the assumptions on ξ^x can be weakened, especially in part (a) of the lemma.
(c) If ν is the Lebesgue measure on \mathbb{R}^d and ξ^x is the Brownian motion in \mathbb{R}^d starting from x , then the process X^0 is the standard Brownian density process.

In [8], part (b) of the lemma is derived from part (a) by means of the time-localization procedure. The main point is that \mathcal{X}^0 is a linear random functional (in the sense of [20]), continuous on $\mathcal{S}(C)$ with the topology induced by $L^q(\tilde{\mathbb{N}})$, where $\tilde{\mathbb{N}}(d\omega) = (1 + |\omega(0)|^2)^p \mathbb{N}(d\omega)$, for some $q \geq 1$ ($q = 4$ turns out to be appropriate). In the terminology of [7], \mathcal{X}^0 is an *admissible* random element of $\mathcal{S}(C)'$. Then \mathcal{X}^0 can be uniquely extended to functionals on C of the form $F_{\varphi,t}(\omega) = \varphi(\omega(t))$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $t \in [0, 1]$, $\omega \in C$ (which do not belong to $\mathcal{S}(C)$, [5]), and we have $\langle X^0(t), \varphi \rangle = \mathcal{X}^0(F_{\varphi,t})$. We say that X^0 is the *time-localization process* of \mathcal{X}^0 .

After this preparation, let us go back to our Cox system described at the beginning of this section. Let $N_n(t)$ be the corresponding empirical process and consider the fluctuation process

$$X_n(t) = \frac{1}{\sqrt{n}}(N_n(t) - \rho_n \nu), \quad t \in [0, 1].$$

Observe that

$$N_n(t) = N_{\rho_n}^0(t), \quad (3.7)$$

where N_r^0 is given by (3.6).

We have the following theorem.

Theorem 3.1. *Let the process ξ^x be of one of the types in Corollary 2.1. Then*

$$X_n \Rightarrow \sqrt{\rho} X^0 \quad \text{in } C([0, 1], \mathcal{S}'(\mathbb{R}^d)),$$

where X^0 is the process defined in Lemma 3.1 (b), independent of ρ .

Proof. It is clear that $\sqrt{\rho} \mathcal{X}^0$ is an admissible random element of $\mathcal{S}(C)'$ and $\sqrt{\rho} X^0$ is its time-localization process in the sense explained above.

Let

$$\mathcal{L} = \text{span} \{ \mathcal{S}(C) \cup \{F_{\varphi,t} : \varphi \in \mathcal{S}(\mathbb{R}^d), t \in [0, 1]\} \}.$$

Recall that

$$\sup_{r>0} E \langle \mathcal{X}_r^0, F \rangle^4 \leq K \langle \tilde{\mathbb{N}}, F^4 \rangle, \quad F \in \mathcal{L}, \quad (3.8)$$

for some constant K (see [7], (4.6) and Theorem 2.4). Fix an $\varepsilon > 0$. By (3.2) there is an $r_0 > 0$ such that

$$P\left(\frac{\rho_n}{n} > r_0\right) < \varepsilon, \quad n = 1, 2, \dots \quad (3.9)$$

For any $F \in \mathcal{L}$, by (3.7) we have

$$\begin{aligned} E(|\mathcal{X}_n(F)| \wedge 1) &\leq \varepsilon + E(|\mathcal{X}_n(F)| \wedge 1) 1_{\{\frac{\rho_n}{n} \leq r_0\}} \\ &= \varepsilon + E\left(\left|\sqrt{\frac{\rho_n}{n}} \mathcal{X}_{\rho_n}^0(F)\right| \wedge 1\right) 1_{\{\frac{\rho_n}{n} \leq r_0\}} \\ &\leq \varepsilon + \sqrt{r_0} (E \mathcal{X}_{\rho_n}^0(F)^4)^{1/4} \\ &\leq \varepsilon + \sqrt{r_0} K \langle \tilde{\mathbb{N}}, F^4 \rangle^{1/4}, \end{aligned}$$

by (3.8), since ρ_n and $\{\mathcal{X}_r^0, r > 0\}$ are independent. This, (3.4) and Proposition 3.1 of [7] imply that the finite-dimensional distributions of X_n converge to those of $\sqrt{\rho}X^0$.

By the theorem of Mitoma [26], to complete the proof it suffices to show that the family $\{\langle X_n, \varphi \rangle, n = 1, 2, \dots\}$ is tight in C for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$. By Lemma 3.1 (b) we know that $\{\langle X_n^0, \varphi \rangle, n = 1, 2, \dots\}$ is tight, and hence the family $\{r\langle X_n^0, \varphi \rangle, r \in [0, r'], n = 1, 2, \dots\}$ is also tight for each $r' > 0$. Given $\varepsilon > 0$, let r_0 satisfy (3.9). Fix a compact subset Γ of C such that

$$P(r\langle X_n^0, \varphi \rangle \notin \Gamma) < \varepsilon \quad \text{for } r \in [0, \sqrt{r_0}], n = 1, 2, \dots \quad (3.10)$$

By (3.7) we have

$$\begin{aligned} P(\langle X_n, \varphi \rangle \notin \Gamma) &= P\left(\sqrt{\frac{\rho_n}{n}}\langle X_{\rho_n}^0, \varphi \rangle \notin \Gamma\right) \\ &\leq \varepsilon + P\left(\sqrt{\frac{\rho_n}{n}}\langle X_{\rho_n}^0, \varphi \rangle \notin \Gamma, \frac{\rho_n}{n} \leq r_0\right) \\ &\leq 2\varepsilon, \end{aligned}$$

by (3.10) and the independence assumption. \square

Remark 3.3.

(a) Remark 2.1 applies to the present case as well, and both Proposition 3.1 and Theorem 3.1 have their counterparts for “charged” particle systems, with the same limits.

(b) Theorem 3.1 was obtained in [12] (in the “charged” particle version) by a different method, in the special case when ξ^x is the Brownian motion starting from x , ν is the Lebesgue measure on \mathbb{R}^d , and the ρ_n are integer-valued.

(c) If the process X^0 satisfies a Langevin equation of the form $dX^0(t) = AX^0(t)dt + dW(t)$, for some $\mathcal{S}'(\mathbb{R}^d)$ -valued Wiener process W (e.g., in the case of the Brownian density process, with $A = \frac{1}{2}\Delta^*$ and $E\langle W(s), \varphi \rangle \langle W(t), \psi \rangle = (s \wedge t) \int \nabla \varphi(x) \cdot \nabla \psi(x) dx$ [1, 4, 24, 30]), then the limit process $X = \sqrt{\rho}X^0$ obviously satisfies $dX(t) = AX(t)dt + \sqrt{\rho}dW(t)$.

4. A Cox system driven by a superprocess

We start with some background on branching particle systems and superprocesses [9, 19]. (In the context of superprocesses the space of measures $\mathcal{M}_p(\dot{\mathbb{R}}^d)$ is used, where $\dot{\mathbb{R}}^d = \mathbb{R}^d \cup \{\tau\}$, τ being an isolated point. This space is not needed in the present paper.)

We consider a system described as follows. At time 0 the particles are distributed according to a Cox random measure with intensity ν , which is a random element of $\mathcal{M}_q(\mathbb{R}^d)$. The particles evolve independently according to the symmetric α -stable process, $\alpha \in (0, 2]$, they branch at rate 1 with a critical $(1 + \beta)$ branching law, $\beta \in (0, 1]$ (whose generating function is $s + (1 + \beta)^{-1}(1 - s)^{1+\beta}$, $s \in [0, 1]$), and the offspring particles obey the same rules. A high-density and small-life limit of this system yields an $\mathcal{M}_q(\mathbb{R}^d)$ -valued Markov process $Y = \{Y(t), t \geq 0\}$, called

the (α, d, β) -superprocess. (The value of q is chosen so that $q > \frac{d}{2}$, and in addition $q < \frac{d+\alpha}{2}$ if $\alpha < 2$). The Laplace functional of $Y(t)$ is given in terms of the Laplace functional of ν by

$$E \exp\{-\langle Y(t), \varphi \rangle\} = E \exp\{-\langle \nu, U_t \varphi \rangle\}, \quad \varphi \in K_q(\mathbb{R}^d)_+; \quad (4.1)$$

$U_t \varphi$ is the unique non-negative solution of the non-linear equation

$$U_t \varphi = T_t \varphi - \gamma \int_0^t T_{t-s} (U_s \varphi)^{1+\beta} ds, \quad (4.2)$$

where $\gamma = 1/(1+\beta)$ and (T_t) is the semigroup of the α -stable process. Note that $Y(0) = \nu$. A basic fact is that for each $t \geq 0$, the empirical measure $N(t)$ of the branching particle system at time t is a Cox random measure whose intensity is $Y(t)$, the state of the superprocess at the same time. (This follows from the form of the Laplace functionals of $N(t)$ and $Y(t)$ and the uniqueness of the solutions of the corresponding log-Laplace equations [11, 15].)

We now consider a modification of the particle system as follows. Let the system evolve as described above until a fixed time t_0 , at which the branching mechanism stops and the particles just go on moving independently. Moreover, at time t_0 the particle motion can change to a different one. Taking t_0 as the new origin of time, we wish to investigate the trajectorial fluctuations of the system as the density of particles tends to infinity. This model may have some physical interest, but our aim here is only to study the trajectorial fluctuations of a Cox system of independent motions which is driven by the non-trivial and interesting random measure $Y(t_0)$. Note that even if the measure ν is deterministic, the initial condition (at time t_0) is of a different type than that of the model in the previous section.

Let the Cox measure that initiates the branching particle system (at time 0) have intensity $n\nu$, n a positive integer, and let Y_n denote the corresponding superprocess. Thus, our Cox system of independent motions is driven by the random measure $Y_n(t_0)$. We must verify the validity of the assumption (iii) of Theorem 2.1, namely $\mu_n/n \Rightarrow \mu$ in $\mathcal{M}_p(\mathbb{R}^d)$. This is a consequence of the following lemma.

Lemma 4.1. *For any $p > q$ and each $t \geq 0$,*

$$\frac{Y_n(t)}{n} \Rightarrow T_t^* \nu \quad \text{in } \mathcal{M}_p(\mathbb{R}^d),$$

where T_t^* is the adjoint of T_t .

The proof of this lemma is an easy combination of Lemma A.2.2 and standard methods in the theory of superprocesses, but we give it for the benefit of a reader not acquainted with the subject.

Proof. We have from (4.1) and (4.2)

$$E \exp\left\{-\left\langle \frac{Y_n(t)}{n}, \varphi \right\rangle\right\} = E \exp\left\{-\left\langle n\nu, U_t \frac{\varphi}{n} \right\rangle\right\}, \quad \varphi \in K_q(\mathbb{R}^d)_+, \quad (4.3)$$

and

$$\left\langle n\nu, U_t \frac{\varphi}{n} \right\rangle = \left\langle n\nu, T_t \frac{\varphi}{n} \right\rangle - \gamma H_n = \langle \nu, T_t \varphi \rangle - \gamma H_n,$$

where

$$H_n = \left\langle n\nu, \int_0^t T_{t-s} \left(U_s \frac{\varphi}{n} \right)^{1+\beta} ds \right\rangle.$$

Since $U_s \varphi \geq 0$ (for $\varphi \geq 0$), it follows from (4.2) that $U_s \varphi \leq T_s \varphi$. Hence

$$\begin{aligned} H_n &\leq \left\langle n\nu, \int_0^t T_{t-s} \left(T_s \frac{\varphi}{n} \right)^{1+\beta} ds \right\rangle \\ &\leq \frac{1}{n^\beta} \left\langle \nu, \int_0^t T_{t-s} \left(T_s \varphi \right)^{1+\beta} ds \right\rangle \\ &\leq \text{const.} \frac{t}{n^\beta} \langle \nu, T_t \varphi \rangle, \end{aligned}$$

and $\langle \nu, T_t \varphi \rangle < \infty$ because $\|T_t \varphi\|_q < \infty$ for $\varphi \in K_q(\mathbb{R}^d)$ [10, 19]. Therefore $H_n \rightarrow 0$, and from (4.3) we have

$$E \exp \left\{ \left\langle \frac{Y_n(t)}{n}, \varphi \right\rangle \right\} \rightarrow E \exp \{ -\langle T_t^* \nu, \varphi \rangle \}, \quad \varphi \in K_q(\mathbb{R}^d)_+$$

The proof is finished by Lemma A.2.2. \square

The trajectorial fluctuation limit follows from Theorem 2.1. The particle motions (after time t_0) are denoted by ξ^x .

Proposition 4.1. *Let the processes ξ^x be of one the types in Corollary 2.1. Then $\mathcal{X}_n \Rightarrow \mathcal{X}$ in $\mathcal{S}(C)'$ and \mathcal{X} is given by*

$$E \exp \{ i \langle \mathcal{X}, F \rangle \} = E \exp \left\{ -\frac{1}{2} \left\langle T_{t_0}^* \nu, \langle M, F^2 \rangle \right\rangle \right\} = L_\nu \left(\frac{1}{2} T_{t_0} \langle M, F^2 \rangle \right), \quad F \in \mathcal{S}(C).$$

Note that

$$T_{t_0} \langle M, F^2 \rangle (x) = \int_{\mathbb{R}^d} p_{t_0}(x, y) E F^2(\xi^y) dy,$$

where p_t is the transition probability density of the α -stable process. If ν is deterministic, then \mathcal{X} is Gaussian.

The main difficulty with the time-localization in this case is that there is not a single admissible measure that works (such as \mathbb{N} in the example of the previous section).

Appendix

A.1. The space $\mathcal{S}(C)'$

We summarize the definitions of $\mathcal{S}(C)$ and $\mathcal{S}(C)'$ [14].

Let $C \equiv C([0, 1], \mathbb{R}^d)$, $C_0 = \{\omega \in C : \omega(0) = 0\}$, λ = the Lebesgue measure on \mathbb{R}^d , W^x = the Wiener measure on C supported on the functions that take the value $x \in \mathbb{R}^d$ at $t = 0$, and \mathbb{W} = the σ -finite Wiener measure on C defined by $d\mathbb{W} = dW^x \lambda(dx)$.

There exists a space \mathcal{H} of test functions on C_0 such that

$$\mathcal{H} \subset L^2(C_0, W^0) \subset \mathcal{H}'$$

is a nuclear Gelfand triple, and the space $\mathcal{S}(C) \equiv \mathcal{S}(\mathbb{R}^d, \mathcal{H})$ of smooth \mathcal{H} -valued functions on \mathbb{R}^d satisfies

$$\mathcal{S}(C) \subset L^2(C, \mathbb{W}) \subset \mathcal{S}(C)',$$

which is also a nuclear Gelfand triple. $\mathcal{S}(C)$ is a nuclear Fréchet space, and since $\mathcal{S}(\mathbb{R}^d, \mathcal{H}) \cong \mathcal{S}(\mathbb{R}^d) \hat{\otimes} \mathcal{H}$, $\mathcal{S}(C)$ can be regarded as a space of test functions on C . Integer powers of elements of $\mathcal{S}(C)$ also belong to $\mathcal{S}(C)$ [22].

We need some precise information on the space \mathcal{H} . For each $j = 0, 1, 2, \dots$, let H_j denote the closure of $\mathcal{D}([0, 1], \mathbb{R}^d)$ (the infinitely differentiable functions from $[0, 1]$ into \mathbb{R}^d that vanish at 0 and 1 together with all their derivatives) with the norm

$$\left(\sum_{r=1}^d \left(\sum_{i=0}^j \int_0^1 (f_r^{(i)}(t))^2 dt \right) \right)^{1/2}, \quad f = (f_1, \dots, f_d).$$

Let $H_j^{\odot n}$ denote the completed symmetric tensor product of H_j with itself n times. An element $F \in L^2(C_0, W^0)$ with Wiener chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$ belongs to \mathcal{H} if and only if

$$\|F\|_{\mathcal{H}_j}^2 := \sum_{n=0}^{\infty} n! \|f_n\|_{H_j^{\odot n}}^2 < \infty$$

for all $j = 0, 1, \dots$. The $\|\cdot\|_{\mathcal{H}_j}$ are seminorms which define the (Fréchet) topology in \mathcal{H} .

A.2. The space $\mathcal{M}_p(\mathbb{R}^d)$ and convergence of random measures

Background for this material can be found in [9, 19].

We write $\langle m, f \rangle = \int f dm$.

$C(E)$ and $\mathcal{M}(E)$ designate the spaces of bounded continuous functions and Radon measures on a topological space E , respectively. $C_c(E)$ is the subspace of elements of $C(E)$ with compact support. The subset of non-negative elements of a function space is denoted with the index “+”. If E is a locally compact metric space, the vague topology on $\mathcal{M}(E)$ is the smallest topology that makes the mappings $\mu \mapsto \langle \mu, \varphi \rangle$ continuous for all $\mu \in C_c(E)$.

For $p > 0$, we define

$$\begin{aligned} \varphi_p(x) &= (1 + |x|^2)^{-p}, \quad x \in \mathbb{R}^d, \\ K_p(\mathbb{R}^d) &= \{\varphi \in C(\mathbb{R}^d) : \varphi = \psi + a\varphi_p, \psi \in C_c(\mathbb{R}^d), a \in \mathbb{R}\}, \\ C_p(\mathbb{R}^d) &= \{\varphi \in C(\mathbb{R}^d) : \lim_{|x| \rightarrow \infty} \varphi(x)/\varphi_p(x) \text{ exists}\}, \\ \mathcal{M}_p(\mathbb{R}^d) &= \{\mu \in \mathcal{M}(\mathbb{R}^d) : \langle \mu, \varphi_p \rangle < \infty\}. \end{aligned}$$

The elements of $\mathcal{M}_p(\mathbb{R}^d)$ are called *p-tempered measures*. Note that the Lebesgue measure λ belongs to $\mathcal{M}_p(\mathbb{R}^d)$ for $p > d/2$. The space $C_p(\mathbb{R}^d)$ is equipped with the norm $\|\varphi\|_p = \sup_x |\varphi(x)|/\varphi_p(x)$.

On $\mathcal{M}_p(\mathbb{R}^d)$ we consider the *p-vague topology*, i.e., the smallest topology that makes the mappings $\mu \mapsto \langle \mu, \varphi \rangle$ continuous for all $\varphi \in C_p(\mathbb{R}^d)$. The space $\mathcal{M}_p(\mathbb{R}^d)$ is Polish. It is easy to see that $K_p(\mathbb{R}^d)$ is $\|\cdot\|_p$ -dense in $C_p(\mathbb{R}^d)$, so the mappings $\mu \mapsto \langle \mu, \varphi \rangle$ are also continuous for $\varphi \in C_p(\mathbb{R}^d)$.

Lemma A.2.1.

(a) For each $\varphi \in C_p(\mathbb{R}^d)_+$, any compact subset of $\mathcal{M}_p(\mathbb{R}^d)$ is contained in $\{\mu : \langle \mu, \varphi \rangle \leq k\}$ for some $k > 0$.

(b) For any $p' > p$ and $k > 0$, the set $\{\mu : \langle \mu, \varphi_p \rangle \leq k\}$ is compact in $\mathcal{M}_{p'}(\mathbb{R}^d)$.

Proof. (a) is obvious. To prove (b), first observe that the set is compact in $\mathcal{M}(\mathbb{R}^d)$ in the vague topology. Hence it suffices to prove that if $\mu_n \rightarrow \mu$ in $\mathcal{M}(\mathbb{R}^d)$ and $\langle \mu_n, \varphi_p \rangle \leq k$ for $n = 1, 2, \dots$, then $\langle \mu_n, \varphi_{p'} \rangle \rightarrow \langle \mu, \varphi_{p'} \rangle < \infty$.

Let $e_m \in C_c(\mathbb{R}^d)_+$, $e_m(x) = 1$ for $|x| \leq m$, $m = 1, 2, \dots$, $e_m \uparrow 1$ as $m \rightarrow \infty$. We have

$$\langle \mu, \varphi_{p'} \rangle = \lim_{m \rightarrow \infty} \langle \mu, \varphi_{p'} e_m \rangle = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle \mu_n, \varphi_{p'} e_m \rangle \leq k.$$

It suffices now to write

$$\begin{aligned} |\langle \mu_n, \varphi_{p'} \rangle - \langle \mu, \varphi_{p'} \rangle| &\leq |\langle \mu_n, \varphi_{p'} \rangle - \langle \mu_n, \varphi_{p'} e_m \rangle| + |\langle \mu_n, \varphi_{p'} e_m \rangle - \langle \mu, \varphi_{p'} e_m \rangle| \\ &\quad + |\langle \mu, \varphi_{p'} e_m \rangle - \langle \mu, \varphi_{p'} \rangle|, \end{aligned}$$

and observe that

$$\begin{aligned} |\langle \mu_n, \varphi_{p'} \rangle - \langle \mu_n, \varphi_{p'} e_m \rangle| &= \left| \int_{|x| > m} \frac{1}{(1 + |x|^2)^{p'-p}} \varphi_p(x) (1 - e_m(x)) \mu_n(dx) \right| \\ &\leq \frac{k}{(1 + m^2)^{p'-p}}. \end{aligned}$$

□

Note that $\{\mu : \langle \mu, \varphi_p \rangle \leq k\}$ is not compact in $\mathcal{M}_p(\mathbb{R}^d)$.

For a random element μ of $\mathcal{M}_p(\mathbb{R}^d)$, the Laplace functional L_μ is defined by

$$L_\mu(\varphi) = E e^{-\langle \mu, \varphi \rangle}, \quad \varphi \in C_p(\mathbb{R}^d)_+.$$

Let $\mu_n, n = 1, 2, \dots$ and μ be random elements of $\mathcal{M}_p(\mathbb{R}^d)$. Convergence in distribution of μ_n to μ as $n \rightarrow \infty$, denoted by $\mu_n \Rightarrow \mu$, is defined (as usual) by $Ef(\mu_n) \rightarrow Ef(\mu)$ as $n \rightarrow \infty$ for all $f \in C(\mathcal{M}_p(\mathbb{R}^d))$.

Lemma A.2.2. If $\mu_n \Rightarrow \mu$ in $\mathcal{M}_p(\mathbb{R}^d)$, then

$$L_{\mu_n}(\varphi) \rightarrow L_\mu(\varphi) \quad \text{as} \quad n \rightarrow \infty \quad (4.4)$$

for all $\varphi \in C_p(\mathbb{R}^d)_+$. Conversely, if (4.4) holds for all $\varphi \in K_p(\mathbb{R}^d)_+$, then $\mu_n \Rightarrow \mu$ in $\mathcal{M}_{p'}(\mathbb{R}^d)$ for any $p' > p$.

Proof. The first statement is clear. It is well known that the distribution of a random measure μ in $\mathcal{M}(\mathbb{R}^d)$ is uniquely determined by its Laplace functional on $C_c(\mathbb{R}^d)_+$ (e.g., [21], Lemma 10.1). Hence, to prove the second statement it suffices to show that the family $\{\mu_n\}_n$ is tight in $\mathcal{M}_{p'}(\mathbb{R}^d)$.

By assumption, $E \exp\{-\langle \mu_n, \varphi_p \rangle k\} \rightarrow E \exp\{-\langle \mu, \varphi_p \rangle k\}$ for all integers $k \geq 0$, and therefore, by the Stone-Weierstrass theorem, $E f(\exp\{-\langle \mu_n, \varphi_p \rangle\}) \rightarrow E f(\exp\{-\langle \mu, \varphi_p \rangle\})$ for all $f \in C([0, 1], \mathbb{R})$. Hence $\exp\{-\langle \mu_n, \varphi_p \rangle\} \Rightarrow \exp\{-\langle \mu, \varphi_p \rangle\}$, and consequently $\langle \mu_n, \varphi_p \rangle \Rightarrow \langle \mu, \varphi_p \rangle$. The tightness of $\{\mu_n\}_n$ now follows by Lemma A.2.1. \square

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