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Functional limit theorems for occupation time fluctuations of branching systems in the case of long-range dependence

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Abstract

A functional central limit theorem is given for the occupation time process of a critical branching particle system in $\mathbb{R}^d$ with symmetric α-stable motion and $\alpha < d < 2\alpha$, which leads to a long-range dependence process called sub-fractional Brownian motion. An analogous result is given for the system without branching and $d < \alpha$, which leads to fractional Brownian motion. A space-time random field approach is used.

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1. Introduction

Consider a system of particles in $\mathbb{R}^d$ starting off from a standard Poisson random field (i.e., with intensity the Lebesgue measure denoted by λ), and independently moving according to a symmetric α-stable Lévy process and undergoing critical binary branching at rate V. As time is accelerated, the rescaled occupation time fluctuations of the system lead in the limit to fractional Brownian motion (fBm) in the case without branching ($V = 0$) and for $d < \alpha$. The covariance function of the fBm is

$$\frac{1}{2}(s^h + t^h - |s-t|^h), \tag{1.1}$$

where $h = 2 - d/\alpha$ ($h \in (1, 3/2]$). In the branching case ($V > 0$) and for $\alpha < d < 2\alpha$ the rescaled occupation time fluctuations of the system lead to sub-fractional Brownian motion (sub-fBm), whose covariance function is

$$s^h + t^h - \frac{1}{2}[(s+t)^h + |s-t|^h], \tag{1.2}$$

where $h = 3 - d/\alpha$ ($h \in (1, 2)$). fBm and sub-fBm exist for all $h \in (0, 2)$ and both processes coincide with Brownian motion (Bm) for $h = 1$. Sub-fBm has some of the main properties of fBm

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but the long-range dependence decays faster, namely, the covariance of increments on intervals at distance τ decays like τ^{h-2} for fBm and like τ^{h-3} for sub-fBm. See Bojdecki et al (2004a) for these and related results. Dzhaparidze and van Zanten (2004) introduced independently a process like sub-fBm in a different context.

The objectives of Bojdecki et al (2004a) were, firstly, to study properties of sub-fBm compared with those of fBm and, secondly, to show that fBm with $h \in (1, 3/2]$ and sub-fBm with $h \in (1, 2)$ are related to occupation time fluctuations of the particle systems described above. Concerning the second objective, convergence of the covariance functions was proved. These results suggest that the occupation time processes of the particle systems have long-range dependence behavior for $V = 0$ and $d < \alpha$, and for $V > 0$ and $\alpha < d < 2\alpha$. However, in order to show that this is so it is necessary to prove much stronger results: functional convergence of the rescaled occupation time fluctuation processes of the particle systems. This is the aim of the present paper. We will prove convergence of the processes in the function space $C([0,\tau], \mathcal{S}'(\mathbb{R}^d))$ for any $\tau > 0$, where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions, dual of the space $\mathcal{S}(\mathbb{R}^d)$ of smooth rapidly decreasing functions. It is well known that this type of space is appropriate for convergence results of this kind due to the nuclear property of $\mathcal{S}(\mathbb{R}^d)$. A consequence of our results is an interesting interpretation of fBm and sub-fBm for $h > 1$. The proof for the branching case combines methods from branching systems (e.g. Dawson et al, 2001, Iscoe, 1986) and a space-time random field approach for convergence of processes in nuclear spaces (Bojdecki et al, 1986). The proof for the non-branching case is similar and simpler.

In a general setting, given a measure-valued process $M = (M(t))_{t\geq 0}$ on $\mathbb{R}^d$, the rescaled occupation time process L_T of M is defined by

$$L_T(t) = \int_0^{Tt} M(s)ds, \quad t \geq 0,$$

where T is a parameter that will tend to infinity, and the fluctuation process X_T of L_T is the signed measure-valued process defined by

$$X_T(t) = \frac{1}{F_T}\int_0^{Tt} (M(s) - EM(s))ds, \quad t \geq 0,$$

where F_T is a norming. One wants to find a suitable F_T such that X_T converges in distribution in $C([0,\tau], \mathcal{S}'(\mathbb{R}^d))$ as $T \to \infty$ for any $\tau > 0$ and identify the limit $\mathcal{S}'(\mathbb{R}^d)$-valued process. In our case the process M is the empirical measure process of the particle system, with or without branching. In the case without branching and Brownian particles ($\alpha = 2$), Deuschel and Wang (1994) proved an occupation time fluctuation result (for a fixed bounded test function with compact support, rather than in the setting of $\mathcal{S}'(\mathbb{R}^d)$-valued processes), by showing tightness and convergence of finite-dimensional distributions, and for this they used fine properties of the motion process (Brownian local time). We do not use local time, and instead of convergence of finite-dimensional distributions we use a space-time random field method developed in Bojdecki et al (1986) for uniqueness and identification of the limit, which is less cumbersome, specially regarding the complexities introduced by the branching.

The space-time method is described as follows. For any continuous $\mathcal{S}'(\mathbb{R}^d)$-valued process $X = (X(t))_{0\leq t\leq\tau}$, let $\widetilde{X}$ denote the random variable in $\mathcal{S}'(\mathbb{R}^{d+1})$ defined by

$$\langle \widetilde{X}, \Phi\rangle = \int_0^\tau \langle X(t), \Phi(\cdot, t)\rangle dt, \quad \Phi \in \mathcal{S}(\mathbb{R}^{d+1}).$$

($\langle\ ,\ \rangle$ denotes duality in the appropriate spaces). It is proved in Bojdecki et al (1986, Theorem 4.3) that if a family of continuous $\mathcal{S}'(\mathbb{R}^d)$-valued processes $\{X_T; T \geq 1\}$ is tight and $\widetilde{X}_T$ converges in distribution in $\mathcal{S}'(\mathbb{R}^{d+1})$ as $T \to \infty$, then $X_T \Rightarrow X$ in $C([0,\tau],\mathcal{S}'(\mathbb{R}^d))$ as $T \to \infty$ for some $\mathcal{S}'(\mathbb{R}^d)$-valued process X ($\Rightarrow$ stands for convergence in distribution). Moreover, $\widetilde{X}_T \Rightarrow \widetilde{X}$ as $T \to \infty$ and the distribution of the process X is determined by that of the random variable $\widetilde{X}$; in particular if $\widetilde{X}$ is Gaussian then so is X (Bojdecki et al, 1986, Theorem 3.4 and Proposition 4.1); see also Bojdecki and Gorostiza (1986, Lemma 3.2). It should be noted that the convergence $\widetilde{X}_T \Rightarrow \widetilde{X}$ in general is not equivalent to the convergence of finite-dimensional distributions.

Another feature of our method is an application of the Feynman-Kac formula (the F-K representation is a useful tool in the analysis of branching systems; see e.g. Dawson, 1993; Gorostiza and Wakolbinger, 1991).

The objective of this paper is to show that the occupation time fluctuations of the particle systems with and without branching have long-range dependence behavior under suitable conditions and d and α. On the other hand, since long-range dependence processes have many areas of application (hydrology, turbulence, communication networks, financial mathematics, etc.), it is worthwhile to study how some of these processes arise from specific stochastic models, and the present ones are examples of this type.

In Section 2 we state the results and in Section 3 we give the proofs.

2. Convergence theorems

Recall that the system consists of particles in $\mathbb{R}^d$ independently evolving according to a symmetric α-stable Lévy process and undergoing critical binary branching at rate V, and starting off at time 0 from a Poisson random field with intensity λ. Let N_t denote the empirical measure of the particle system at time t, i.e., $N_t(A)$ is the number of particles in the set $A \subset \mathbb{R}^d$ at time t. Sometimes we write $\langle \mu, f\rangle = \int f d\mu$ where μ is a measure and f a measurable function (recall that $\langle\ ,\ \rangle$ is also used for the pairing of spaces in duality).

For $T > 0$, let $(L_T(t))_{t\geq 0}$ denote the rescaled occupation time process of the process $N = (N_t)_{t\geq 0}$:

$$\langle L_T(t),\varphi\rangle = \int_0^{Tt} \langle N_s,\varphi\rangle ds = T\int_0^t \langle N_{Ts},\varphi\rangle ds, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Due to the uniform Poisson initial condition and the criticality of the branching, we have $E\langle L_T(t),\varphi\rangle = Tt\langle\lambda,\varphi\rangle$ for both the branching and the non-branching cases. Then the occupation time fluctuation process $(X_T(t))_{t\geq 0}$ is given by

$$\langle X_T(t),\varphi\rangle = \frac{1}{F_T}(\langle L_T(t),\varphi\rangle - Tt\langle\lambda,\varphi\rangle) = \frac{T}{F_T}\int_0^t (\langle N_{Ts},\ \varphi\rangle - \langle\lambda,\varphi\rangle)ds, \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.1}$$

Theorem 2.1. *For the system without branching* ($V = 0$), $d < \alpha$ *and* $F_T = T^{1-d/2\alpha}$, *we have* $X_T \Rightarrow X$ *in* $C([0,\tau],\mathcal{S}'(\mathbb{R}^d))$ *as* $T \to \infty$ *for any* $\tau > 0$, *where* $(X(t))_{t\geq 0}$ *is a centered Gaussian process with covariance function*

$$Cov(\langle X(s),\varphi\rangle,\langle X(t),\psi\rangle) = \langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle\frac{\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}(s^h + t^h - |s-t|^h),$$
$$\varphi,\psi \in \mathcal{S}(\mathbb{R}^d), \tag{2.2}$$

with $h = 2 - d/\alpha$.

Theorem 2.2. *For the branching system* ($V > 0$), $\alpha < d < 2\alpha$ *and* $F_T = T^{(3-d/\alpha)/2}$, *we have* $X_T \Rightarrow X$ *in* $C([0,\tau],\mathcal{S}'(\mathbb{R}^d))$ *as* $T \to \infty$ *for any* $\tau > 0$, *where* $(X(t))_{t\geq 0}$ *is a centered Gaussian* *process with covariance function*

$$
\begin{aligned}
&Cov(\langle X(s),\varphi\rangle,\langle X(t),\psi\rangle)\\
&= \langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle\frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}\left(s^h+t^h-\frac{1}{2}[(s+t)^h+|s-t|^h]\right),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi,\psi\in\mathcal{S}(\mathbb{R}^d), \qquad (2.3)
\end{aligned}
$$

with $h = 3 - d/\alpha$.

Remark 2.3 (a) The limit processes above can be represented as follows: For Theorem 2.1,

$$X = \left(\frac{\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}\right)^{1/2}\lambda\xi^h$$

where $\xi^h = (\xi_t^h)_{t\geq 0}$ is fBm, i.e., a real, continuous, centered Gaussian process with covariance (1.1), and for Theorem 2.2,

$$X = \left(\frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}\right)^{1/2}\lambda\beta^h,$$

where $\beta^h = (\beta_t^h)_{t\geq 0}$ is sub-fBm, i.e., a real, continuous, centered Gaussian process with covariance (1.2). In both cases the limit process X has a trivial spatial structure (Lebesgue measure), whereas the time structure is complicated, with long-range dependence. The processes ξ^h and β^h are non-Markovian (the covariances do not have the triangular property, e.g. Neveu, 1968), and the $\mathcal{S}'(\mathbb{R}^d)$-valued processes with covariance (2.2) and (2.3) are non-Markovian as well (see Fernández, 1990, condition (M)).
(b) A careful analysis of the proofs below shows that in the application of the space-time method the space $\mathcal{S}(\mathbb{R}^{d+1})$ can be replaced by $\mathcal{D}(\mathbb{R})\widehat{\otimes}\mathcal{S}(\mathbb{R}^d)$, so Theorem 7.1 of Bojdecki et al (1986) can be used to obtain weak convergence of X_T in $C([0,\infty),\mathcal{S}'(\mathbb{R}^d))$ in both theorems.
(c) We have already mentioned that a result like Theorem 2.1 was proved with a different approach by Deuschel and Wang (1994, Theorem 0.4 (i)) for $d = 1$, $\alpha = 2$ (the Brownian case) and a fixed bounded φ (not necessarily smooth) with compact support. A result like Theorem 2.2 is stated without proof in Iscoe (1986, Theorem 6.2) for the case $d = 3$, $\alpha = 2$, in the context of superprocesses, which is simpler than that of branching particle systems. Hong (2004) also considered the superprocess case with general α, and proved the convergence of finite-dimensional distributions of real processes (for a fixed test function), but not the tightness. None of these works refer to long-range dependence.
(d) The condition $d < \alpha$ in Theorem 2.1 corresponds to strict recurrence of the particle motion, and the condition $\alpha < d < 2\alpha$ in Theorem 2.2 corresponds to strict weak transience of the particle motion. In the cases $d \geq \alpha$ for the non-branching system and $d \geq 2\alpha$ for the branching system, the limit covariances of the occupation time fluctuation processes (with appropriate normings F_T) indicate that the limit $\mathcal{S}'(\mathbb{R}^d)$-valued processes do not have long-range dependence. Our functional convergence results for these cases will appear in Bojdecki et al (2004b). In these cases the tightness proofs require more work. It should be noted that convergence of the covariance alone does not always yield reliable information. It can be shown that in the case $d < \alpha$ for the branching system with norming $F_T = T^{(3-d/\alpha)/2}$ the covariance has a non-trivial limit, whereas $\langle X_T,\varphi\rangle \to 0$ in $L^1(\Omega)$ as $T\to\infty$, $\varphi\in\mathcal{S}(\mathbb{R}^d)$.
(e) It was noted in Bojdecki et al (2004a) that if the branching particle system starts off from equilibrium (instead of Poisson random field), then the rescaled occupation time fluctuation

process (with $\alpha < d < 2\alpha$ and $F_T = T^{(3-d/\alpha)/2}$) has a limit covariance which essentially coincides with the case of the system without branching (with $d < \alpha$ and $F_T = T^{(1-d/2\alpha)}$), i.e., the temporal structure is fBm. We think that a functional limit theorem also holds for the branching system in equilibrium but we have not attempted to prove it.

(f) The reason for the long-range dependence in the system without branching is that, by recurrence of the particle motion, each particle enters into any given interval infinitely often and at arbitrarily large times, each time adding a random amount to the occupation time of the interval. Based on the following knowledge, we think that an analogous interpretation holds for the long-range dependence in the branching system. In the Brownian case ($\alpha = 2$) and in equilibrium, "clan recurrence" occurs in dimensions $d = 3$ and 4 (a "clan" is a family of particles with eventually backwards coalescing paths). This means that any given ball is entered into by clans infinitely often and at arbitrarily large times, each time adding a random amount to the occupation time of the ball (see Stöckl and Wakolbinger, 1994, Theorem 1; and in the context of super-Brownian motion, Dawson and Perkins, 1999). Since the branching system started off from Poisson approaches equilibrium, intuitively it seems that a similar effect happens for large time, and also for general α with $\alpha < d \leq 2\alpha$. However, this question remains to be studied rigorously, in particular the loss of the long-range dependence at the borderline $d = 2\alpha$.

3. Proofs

We will prove only Theorem 2.2 because the proof of Theorem 2.1 goes along the same lines without the complexities of the branching.

The covariance function of the empirical process N of the particle system described above is given by

$$Cov(\langle N_u, \varphi\rangle, \langle N_v, \psi\rangle)$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\varphi}(z)\overline{\widehat{\psi}(z)} \left(e^{-(v-u)|z|^\alpha} + V \int_0^u e^{-(u+v-2r)|z|^\alpha} dr \right) dz, \quad u \leq v, \tag{3.1}$$

where $\widehat{\varphi}(z) = \int_{\mathbb{R}^d} e^{ix\cdot z}\varphi(x)dx$ is the Fourier transform (Bojdecki et al, 2004a).

To simplify notation and with no loss of generality we assume $\tau = 1$. We denote by C, C_1, etc. generic positive constants, putting possible dependencies in parenthesis.

The result will be established in several steps.

1. Tightness.

By Mitoma's theorem (Mitoma, 1983), in order to prove tightness of $\{X_T; T \geq 1\}$ in $C([0,1], \mathcal{S}'(\mathbb{R}^d))$ it suffices to show tightness of the real processes $\{\langle X_T, \varphi\rangle; T \geq 1\}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We will prove

$$E(\langle X_T(t), \varphi\rangle - \langle X_T(s), \varphi\rangle)^2 \leq C(\varphi, d, \alpha)|t-s|^h, \quad s, t \in [0,1]. \tag{3.2}$$

(recall that $h = 3 - d/\alpha > 1$). The result will then follow by Billingsley (1968, Theorem 12.3), since $X_T(0) = 0$.

From (2.1) and (3.1) we have, for $s \leq t$,

$$E(\langle X_T(t), \varphi\rangle - \langle X_T(s), \varphi\rangle)^2 = \frac{T^2}{F_T^2} \int_s^t \int_s^t Cov(\langle N_{Tu}, \varphi\rangle, \langle N_{Tv}, \varphi\rangle) dudv = I + J,$$

where (since $F_T = T^{(3-d/\alpha)/2}$)

$$\begin{aligned} I &= \frac{2}{(2\pi)^d} T^{d/\alpha - 1} \int_s^t \int_s^v \int_{\mathbb{R}^d} |\widehat{\varphi}(z)|^2 e^{-T(v-u)|z|^\alpha} dz du dv, \\ J &= V \frac{2}{(2\pi)^d} T^{d/\alpha} \int_s^t \int_s^v \int_{\mathbb{R}^d} |\widehat{\varphi}(z)|^2 \int_0^u e^{-T(u+v-2r)|z|^\alpha} dr dz du dv. \end{aligned}$$

To estimate I we write

$$\begin{aligned} &\int_s^t \int_s^v e^{-T(v-u)|z|^\alpha} du dv = \frac{1}{T|z|^\alpha} \int_s^t (1 - e^{-Tv|z|^\alpha} e^{Ts|z|^\alpha}) dv \\ &= \frac{1}{T|z|^\alpha} \int_0^{t-s} (1 - e^{-Tv|z|^\alpha}) dv \ \leq \ \frac{1}{T|z|^\alpha} \int_0^{t-s} (T|z|^\alpha v)^{2-d/\alpha} dv \\ &= \frac{C(d,\alpha)}{|z|^{d-\alpha}} T^{1-d/\alpha} (t-s)^h, \end{aligned}$$

where we have used the obvious inequality $1 - e^{-x} \leq x^\delta$, valid for all $x > 0, 0 < \delta \leq 1$ (note that $0 < 2 - d/\alpha < 1$). Hence we obtain

$$I \leq C_1(d,\alpha) \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^{d-\alpha}} dz (t-s)^h = C_1(\varphi, d, \alpha)(t-s)^h.$$

Next, since $\widehat{\varphi}$ is bounded we have

$$J \leq C(\varphi) T^{d/\alpha} \int_s^t \int_s^v \int_{\mathbb{R}^d} \int_0^u e^{-T(u+v-2r)|z|^\alpha} dr dz du dv.$$

Substituting $z = (T(u+v-2r))^{-1/\alpha} y$ and observing that

$$\int_0^u (u+v-2r)^{-d/\alpha} dr \leq \frac{1}{2(d/\alpha - 1)} (v-u)^{1-d/\alpha} \quad \text{for } \alpha < d, u < v,$$

we obtain

$$J \leq C(\varphi) \int_{\mathbb{R}^d} e^{-|y|^\alpha} dy \int_s^t \int_s^v (v-u)^{1-d/\alpha} u du dv \leq C(\varphi, d, \alpha)(t-s)^h.$$

Hence (3.2) is proved.

2. Space-time method.

For brevity, denote by $C_h(s,t)$ the covariance function (1.2) of the sub-fBm β^h, i.e.,

$$C_h(s,t) = s^h + t^h - \frac{1}{2}[(s+t)^h + |s-t|^h], \tag{3.3}$$

where $h \in (0,2)$ (see Bojdecki et al, 2004, for existence of β^h).

Let

$$\langle \widetilde{X}_T, \Phi \rangle = \int_0^1 \langle X_T(t), \Phi(\cdot, t) \rangle dt, \ \Phi \in \mathcal{S}(\mathbb{R}^{d+1}), \tag{3.4}$$

where $X_T(t)$ is given by (2.1). As explained in the Introduction, since we already have the tightness, the theorem will be proved by showing that $\widetilde{X}_T \Rightarrow \widetilde{X}$ as $T \to \infty$, where

$$\langle \widetilde{X}, \Phi \rangle = \int_0^1 \langle X(t), \Phi(\cdot, t)\rangle dt, \ \ \Phi \in \mathcal{S}(\mathbb{R}^{d+1}),$$

and $X = (X(t))_{t\geq 1}$ is the Gaussian process with the covariance given by (2.3). By the nuclear property of $\mathcal{S}(\mathbb{R}^{d+1})$ (so Lévy's continuity theorem holds) it suffices to prove that

$$\langle \widetilde{X}_T, \Phi \rangle \Rightarrow K \int_0^1 \int_{\mathbb{R}^d} \Phi(x,t) dx \beta_t^h dt \tag{3.5}$$

as $T \to \infty$ for any $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$, where

$$K = \left(\frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)} \right)^{1/2}. \tag{3.6}$$

The convergence (3.5) will be established if we prove that for any non-negative $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$,

$$\begin{aligned} &\lim_{T\to\infty} E\mathrm{exp}\{-\langle \widetilde{X}_T, \Phi\rangle\} = E\mathrm{exp}\left\{ -K \int_0^1 \int_{\mathbb{R}^d} \Phi(x,t)dx\beta_t^h dt \right\} \\ &= \exp\left\{ \frac{K^2}{2} \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(x,t)\Phi(y,s) dx dy C_h(s,t) ds dt \right\} \end{aligned} \tag{3.7}$$

with $C_h(s,t)$ given by (3.3). Indeed, it is known that if ζ_n are real random variables such that

$$\lim_{n\to\infty} E\mathrm{exp}\{-\theta\zeta_n\} = \exp\left\{ \frac{1}{2}\sigma^2\theta^2 \right\}$$

for each $\theta > 0$, then $\zeta_n \Rightarrow N(0, \sigma^2)$ as $n \to \infty$, and the same holds for multidimensional random variables. So (3.7) implies (3.5) for any non-negative $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$. To obtain (3.5) for general $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$ it suffices to apply the two-dimensional version of the result above and use the following simple lemma:

Lemma. *Any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be written as $\varphi = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d), \varphi_1, \varphi_2 \geq 0$.*

Proof of the Lemma. There exists $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $|\varphi| \leq \psi$ (Bojdecki and Gorostiza, 1999, Lemma 2.6). Now put $\varphi_1 = \psi + \varphi, \varphi_2 = \psi$. □

Note that Iscoe (1986) in his argument, not using this lemma, has to work with a space larger than $\mathcal{S}(\mathbb{R}^d)$ ($\mathcal{S}(\mathbb{R}^{d+1})$ in our case), containing also non-smooth functions. In our case it is convenient (though probably not absolutely necessary) to remain in $\mathcal{S}(\mathbb{R}^{d+1})$ since then the Fourier transforms also belong to (complex) $\mathcal{S}(\mathbb{R}^{d+1})$.

3. Computing $E\mathrm{exp}\{-\langle \widetilde{X}_T, \Phi\rangle\}$

Substituting (2.1) in (3.4) and interchanging orders of integration we obtain

$$\langle \widetilde{X}_T, \Phi \rangle = \frac{T}{F_T} \left[\int_0^1 \langle N_{Ts}, \Psi(\cdot, s)\rangle ds - \left\langle \lambda, \int_0^1 \Psi(\cdot, s) ds \right\rangle \right], \tag{3.8}$$

where

$$\Psi(x,s)=\int_s^1 \Phi(x,t)dt. \tag{3.9}$$

By the Poisson initial condition we have, from (3.8),

$$\begin{aligned}&E\exp\{-\langle \widetilde{X}_T,\Phi\rangle\}\\ &=\exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,s)dsdx\right\}\exp\left\{\int_{\mathbb{R}^d}\left(E\exp\left\{-\int_0^T\langle N_s^x,\Psi_T(\cdot,s)\rangle ds\right\}-1\right)dx\right\},\end{aligned} \tag{3.10}$$

where

$$\Psi_T(x,s)=\frac{1}{F_T}\Psi\left(x,\frac{s}{T}\right), \tag{3.11}$$

and N_s^x denotes the empirical measure of the particle system with initial condition $N_0^x=\delta_x$.

For any $\Psi\in\mathcal{S}(\mathbb{R}^{d+1})$, $\Psi\geq 0$, let

$$w(x,r,t)\equiv w_\Psi(x,r,t)=E\exp\left\{-\int_0^t\langle N_s^x,\Psi(\cdot,r+s)\rangle ds\right\},\ x\in\mathbb{R}^d,\quad r,t\geq 0. \tag{3.12}$$

The reason for the apparently superfluous variable r will be seen later. We have $0\leq w(x,r,t)\leq 1$ and $w(x,r,0)=1$.

Let $\zeta^x=(\zeta_t^x)_{t\geq 0}$ denote the symmetric α-stable process on $\mathbb{R}^d$ such that $\zeta_0^x=x$.

By a renewal argument (conditioning on the time of the first branching) and using the fact that the branching is binary critical (0 or 2 particles with probability 1/2 each case) we obtain

$$\begin{aligned} w(x,r,t) &= e^{-Vt}E\exp\left\{-\int_0^t\Psi(\zeta_s^x,r+s)ds\right\}\\ &+ V\int_0^t e^{-Vs}E\exp\left\{-\int_0^s\Psi(\zeta_u^x,r+u)du\right\}\frac{1}{2}\left[1+w^2(\zeta_s^x,r+s,t-s)\right]ds,\end{aligned}$$

(the generating function of the branching law is $\theta\mapsto\frac{1}{2}(1+\theta^2)$). Hence

$$w(x,r,t)=e^{-Vt}h(x,r,t)+Ve^{-Vt}\int_0^t e^{Vs}k(x,r,t-s)ds, \tag{3.13}$$

where

$$h(x,r,t)\equiv h_\Psi(x,r,t)=E\exp\left\{-\int_0^t\Psi(\zeta_s^x,r+s)ds\right\} \tag{3.14}$$

and

$$k(x,r,\sigma)\equiv k_\Psi(x,r,\sigma)=E\exp\left\{-\int_0^\sigma\Psi(\zeta_u^x,r+u)du\right\}\frac{1}{2}[1+(w(\zeta_\sigma^x,r+\sigma,s))^2]. \tag{3.15}$$

(Note that $k(x,r,\sigma)$ also depends on s, but s is treated as a fixed parameter here, so it is omitted).

By the Feynman-Kac formula, $h(x,r,t)$ given by (3.14) satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}h(x,r,t) &= \left(\Delta_\alpha+\frac{\partial}{\partial r}-\Psi(x,r)\right)h(x,r,t),\\ h(x,r,0) &= 1,\end{aligned} \tag{3.16}$$

and $k(x,r,\sigma)$ given by (3.15) satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial\sigma}k(x,r,\sigma) &= \left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi(x,r)\right) k(x,r,\sigma), \\ k(x,r,0) &= \frac{1}{2}[1+(w(x,r,t))^2],\end{aligned} \tag{3.17}$$

where $\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$ is the infinitesimal generator of the α-stable process. These equations are understood in the mild (integral) sense.

Now, differentiating (3.13) with respect to t and using (3.16) and (3.17) we have

$$\begin{aligned}&\frac{\partial}{\partial t}w(x,r,t) = -Ve^{-Vt}h(x,r,t) + e^{-Vt}\frac{\partial}{\partial t}h(x,r,t) \\ &-V^2e^{-Vt}\int_0^t e^{Vs}k(x,r,t-s)ds + V\frac{1}{2}[1+(w(x,r,t))^2] + Ve^{-Vt}\int_0^t e^{Vs}\frac{\partial}{\partial t}k(x,r,t-s)ds \\ &= -Vw(x,r,t) + V\frac{1}{2}[1+(w(x,r,t))^2] + e^{-Vt}\left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi(x,r)\right)h(x,r,t) \\ &+Ve^{-Vt}\int_0^t e^{Vs}\left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi(x,r)\right)k(x,r,t-s)ds,\end{aligned}$$

so $w(x,r,t)$ satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}w(x,r,t) &= \left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi(x,r)\right)w(x,r,t) + \frac{V}{2}[1-w(x,r,t)]^2, \\ w(x,r,0) &= 1.\end{aligned} \tag{3.18}$$

Let

$$v(x,r,t) \equiv v_\Psi(x,r,t) = 1 - w_\Psi(x,r,t). \tag{3.19}$$

Then $0 \le v(x,r,t) \le 1$, and by (3.18) $v(x,r,t)$ satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}v(x,r,t) &= \left(\Delta_\alpha + \frac{\partial}{\partial r}\right)v(x,r,t) + \Psi(x,r)(1-v(x,r,t)) - \frac{V}{2}(v(x,r,t))^2, \\ v(x,r,0) &= 0,\end{aligned} \tag{3.20}$$

whose integral version is

$$v(x,r,t) = \int_0^t \mathcal{T}_{t-s}\left[\Psi(\cdot,r+t-s)(1-v(\cdot,r+t-s,s)) - \frac{V}{2}(v(\cdot,r+t-s,s))^2\right](x)ds, \tag{3.21}$$

where $(\mathcal{T}_t)_{t\ge 0}$ is the semigroup of the α-stable process. Hence

$$\int_{\mathbb{R}^d} v(x,r,t)dx = \int_0^t\int_{\mathbb{R}^d}\left[\Psi(x,r+t-s)(1-v(x,r+t-s,s)) - \frac{V}{2}(v(x,r+t-s,s))^2\right]dxds. \tag{3.22}$$

Now we return to (3.10) with Ψ_T defined by (3.9) and (3.11). Using (3.12), (3.19) we obtain

$$E\exp\{-\langle \widetilde{X}_T, \Phi\rangle\} = \exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,s)dsdx\right\}\exp\left\{-\int_{\mathbb{R}^d} v_{\Psi_T}(x,0,T)dx\right\},$$

and by (3.22) we have finally

$$
\begin{aligned}
&E\exp\{-\langle \widetilde{X}_T,\Phi\rangle\}\\
&=\exp\left\{\int_0^T\int_{\mathbb{R}^d}\Psi_T(x,T-s)v_{\Psi_T}(x,T-s,s)dxds+\frac{V}{2}\int_0^T\int_{\mathbb{R}^d}(v_{\Psi_T}(x,T-s,s))^2dxds\right\}. \quad (3.23)
\end{aligned}
$$

4. Obtaining $\lim_{T\to\infty} E\exp\{-\langle \widetilde{X}_T,\Phi\rangle\}$

We rewrite (3.23) as

$$
E\exp\{-\langle \widetilde{X}_T,\Phi\rangle\}=\exp\left\{\frac{V}{2}(I_1(T)+I_2(T))+I_3(T))\right\}, \tag{3.24}
$$

where

$$
\begin{aligned}
I_1(T) &= \int_0^T\int_{\mathbb{R}^d}\left(\int_0^s \mathcal{T}_u\Psi_T(\cdot,T+u-s)(x)du\right)^2 dxds, &(3.25)\\
I_2(T) &= \int_0^T\int_{\mathbb{R}^d}\left[(v_{\Psi_T}(x,T-s,s))^2-\left(\int_0^s \mathcal{T}_{s-u}\Psi_T(\cdot,T-u)du\right)^2\right]dxds, &(3.26)\\
I_3(T) &= \int_0^T\int_{\mathbb{R}^d}\Psi_T(x,T-s)v_{\Psi_T}(x,T-s,s)dxds. &(3.27)
\end{aligned}
$$

We will prove the following limits:

$$
\begin{aligned}
I_1(T) &\to \frac{K^2}{V}\int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(x,t)\Phi(y,s)dxdyC_h(s,t)dsdt, &(3.28)\\
I_2(T) &\to 0, &(3.29)\\
I_3(T) &\to 0, &(3.30)
\end{aligned}
$$

as $T\to\infty$, which yields (3.7). This scheme is analogous to that of Iscoe (1986, proof of Theorem 5.4, which deals with a single time). The present case is more intricate because the non-linear equations for the particle system are more complicated than those of superprocesses and we are working with all the times. On the other hand, our use of Fourier transform provides a simpler way of handling the time variable.

For simplicity, we prove (3.28)-(3.30) for functions Φ of the form $\Phi(x,t)=\varphi(x)\psi(t),\varphi\in\mathcal{S}(\mathbb{R}^d),\psi\in\mathcal{S}(\mathbb{R})$, $\varphi,\psi\geq 0$. For general Φ the proofs are similar, only notation is more cumbersome. Let

$$
\chi(t)=\int_t^1\psi(s)ds,\ \ \chi_T(t)=\chi\left(\frac{t}{T}\right). \tag{3.31}
$$

Proof of (3.28):

From (3.25) and (3.31),

$$
\begin{aligned}
I_1(T) &= \frac{1}{F_T^2}\int_0^T\int_{\mathbb{R}^d}\left(\int_0^{T-s}\mathcal{T}_u\varphi(x)\chi_T(u+s)du\right)^2dxds\\
&= \frac{1}{F_T^2}\int_0^T\int_{\mathbb{R}^d}\left(\int_s^T\mathcal{T}_{u-s}\varphi(x)\chi_T(u)du\right)^2dxds
\end{aligned}
$$

$$= \frac{1}{F_T^2}\int_0^T\int_s^T\int_s^T\int_{\mathbb{R}^d}\mathcal{T}_{u-s}\varphi(x)\mathcal{T}_{v-s}\varphi(x)dx\chi_T(u)\chi_T(v)dudvds$$

(Plancherel formula and $\widehat{\mathcal{T}_t\varphi}(z)=e^{-t|z|^\alpha}\widehat{\varphi}(z)$)

$$= \frac{1}{(2\pi)^dF_T^2}\int_0^T\int_s^T\int_s^T\int_{\mathbb{R}^d}e^{-(u+v-2s)|z|^\alpha}|\widehat{\varphi}(z)|^2dz\chi_T(u)\chi_T(v)dudvds$$

$$= \frac{T}{(2\pi)^dF_T^2}\int_0^1\int_{sT}^T\int_{sT}^T\int_{\mathbb{R}^d}e^{-(u+v-2sT)|z|^\alpha}|\widehat{\varphi}(z)|^2dz\chi_T(u)\chi_T(v)dudvds$$

$$= \frac{T^3}{(2\pi)^dF_T^2}\int_0^1\int_s^1\int_s^1\int_{\mathbb{R}^d}e^{-T(u+v-2s)|z|^\alpha}|\widehat{\varphi}(z)|^2dz\chi(u)\chi(v)dudvds$$

$$= \frac{2}{(2\pi)^d}\frac{T^3}{F_T^2}\int_0^1\int_0^u\int_0^v\int_{\mathbb{R}^d}e^{-T(u+v-2s)|z|^\alpha}|\widehat{\varphi}(z)|^2dz\chi(u)\chi(v)dsdvdu$$

(let $z=(T(u+v-2s))^{-1/\alpha}y$)

$$= \frac{2}{(2\pi)^d}\frac{T^{3-d/\alpha}}{F_T^2}\int_0^1\int_0^u\int_0^v\int_{\mathbb{R}^d}e^{-|y|^\alpha}|\widehat{\varphi}(y/(T(u+v-2s))^{1/\alpha})|^2dy$$
$$\cdot(u+v-2s)^{-d/\alpha}\chi(u)\chi(v)dsdvdu$$

($F_T=T^{(3-d/\alpha)/2}, h=3-d/\alpha$)

$$\underset{T\to\infty}{\longrightarrow} \frac{2}{(2\pi)^d}|\widehat{\varphi}(0)|^2\int_{\mathbb{R}^d}e^{-|y|^\alpha}dy\int_0^1\int_0^u\int_0^v(u+v-2s)^{h-3}ds\chi(u)\chi(v)dvdu$$

$$= \langle\lambda,\varphi\rangle^2\frac{\Gamma(3-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)(2-h)}\int_0^1\int_0^u[(u-v)^{h-2}-(u+v)^{h-2}]\chi(u)\chi(v)dvdu$$

$$= \langle\lambda,\varphi\rangle^2\frac{\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)}\int_0^1\int_0^u[(u-v)^{h-2}-(u+v)^{h-2})]\chi(u)\chi(v)dvdu, \quad (3.32)$$

which is finite since $1<h<2$ ($\alpha<d<2\alpha$).

On the other hand, by (3.3) the exponent on the right hand side of (3.7) is given by

$$K^2\langle\lambda,\varphi\rangle^2\int_0^1\int_0^t\left(s^h+t^h-\frac{1}{2}[(s+t)^h+(t-s)^h]\right)\psi(t)\psi(s)dsdt, \quad (3.33)$$

and it is a calculus exercise, integrating by parts several times, to show that

$$\int_0^1\int_0^u\left(u^h+v^h-\frac{1}{2}[(u+v)^h+(u-v)^h]\right)\psi(u)\psi(v)dvdu$$
$$=\frac{1}{2}h(h-1)\int_0^1\int_0^u[(u-v)^{h-2}-(u+v)^{h-2}]\chi(v)\chi(u)dvdu, \quad (3.34)$$

which together with (3.6) proves that (3.33) equals the limit in (3.32) multiplied by V.

Proof of (3.29):

Let

$$n(x,r,s)\equiv n_\Psi(x,r,s)=\int_0^s\mathcal{T}_{s-u}\Psi(\cdot,r+s-u)(x)du. \quad (3.35)$$

By (3.21) we have

$$v(x,r,s)\le n(x,r,s). \quad (3.36)$$

Hence, from (3.27) and (3.35) (with $r=T-s$, then put $u=s$)

$$0\le -I_2(T)=\int_0^T\int_{\mathbb{R}^d}[(n_T(x,T-s,s))^2-(v_T(x,T-s,s))^2]dxds, \tag{3.37}$$

where $n_T\equiv n_{\Psi_T}, v_T\equiv v_{\Psi_T}$. Now, from (3.21) and (3.36),

$$\begin{aligned}
&0\le n_T(x,T-s,s)-v_T(x,T-s,s)\\
&\quad= \int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot,T-u)v_T(\cdot,T-u,u)+\frac{V}{2}(v_T(\cdot,T-u,u))^2\right](x)du\\
&\quad\le \int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot,T-u)n_T(\cdot,T-u,u)+\frac{V}{2}(n_T(\cdot,T-u,u))^2\right](x)du,
\end{aligned} \tag{3.38}$$

and

$$n_T(x,T-s,s)+v_T(x,T-s,s)\le 2n_T(x,T-s,s)=2\int_0^s\mathcal{T}_{s-u}\Psi_T(\cdot,T-u)(x)du. \tag{3.39}$$

Hence (since $n^2-v^2=(n-v)(n+v)$)

$$-I_2(T)\le 2J_1(T)+VJ_2(T), \tag{3.40}$$

where, by (3.38) and (3.39),

$$J_1(T) = \int_0^T\int_{\mathbb{R}^d}\int_0^s\mathcal{T}_{s-u}[\Psi_T(\cdot,T-u)n_T(\cdot,T-u,u)](x)du\int_0^s\mathcal{T}_u\Psi_T(\cdot,T-s+u)(x)dudxds, \tag{3.41}$$

$$J_2(T) = \int_0^T\int_{\mathbb{R}^d}\int_0^s\mathcal{T}_{s-u}(n_T(\cdot,T-u,u))^2(x)\int_0^s\mathcal{T}_u\Psi_T(\cdot,T-s+u)(x)dudxds. \tag{3.42}$$

We will show that $J_1(T)$ and $J_2(T)\to 0$ as $T\to\infty$, which proves (3.29). Again we consider for simplicity Φ of the form $\Phi(x,t)=\varphi(x)\psi(t)$.

Since χ and $\widehat{\varphi}$ are bounded, then, from (3.41) and (3.31), and using the formula

$$\begin{aligned}
\int_{\mathbb{R}^d}\mathcal{T}_{s-u}[\varphi(\cdot)\mathcal{T}_r\varphi(\cdot)](x)\mathcal{T}_{u'}\varphi(x)dx &= \int_{\mathbb{R}^d}\varphi(x)\mathcal{T}_r\varphi(x)\mathcal{T}_{s+u'-u}\varphi(x)dx\\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\widehat{\mathcal{T}_r\varphi}(z)\widehat{\mathcal{T}_{s+u'-u}\varphi}(z')\overline{\widehat{\varphi}(z+z')}dzdz',
\end{aligned}$$

which follows from the identity

$$\int_{\mathbb{R}^d}\varphi_1(x)\varphi_2(x)\mu(dx)=\frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\widehat{\varphi}_1(z)\widehat{\varphi}_2(z')\overline{\widehat{\mu}(z+z')}dzdz'$$

applied to the finite measure $d\mu=\varphi(x)dx$, we have

$$\begin{aligned}
&J_1(T)\le\frac{C}{F_T^3}\int_0^T\int_{\mathbb{R}^d}\int_0^s\mathcal{T}_{s-u}\left[\varphi(\cdot)\int_0^u\mathcal{T}_r\varphi(\cdot)dr\right](x)du\int_0^s\mathcal{T}_{u'}\varphi(x)du'dxds\\
&\le\frac{C_1}{F_T^3}\int_0^T\int_{\mathbb{R}^d}|\widehat{\varphi}(z)||\widehat{\varphi}(z')||\widehat{\varphi}(z+z')|e^{-s|z'|^\alpha}\int_0^s e^{u|z'|^\alpha}\int_0^u e^{-r|z|^\alpha}dr\int_0^s e^{-u'|z'|^\alpha}du'dudzdz'ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C_2}{F_T^3}\int_0^T \int_{\mathbb{R}^{2d}} |\widehat{\varphi}(z)\ | e^{-s|z'|^\alpha} \int_0^s \frac{e^{u|z'|^\alpha} - e^{u(|z'|^\alpha - |z|^\alpha)}}{|z|^\alpha} \frac{1-e^{-s|z'|^\alpha}}{|z'|^\alpha} dudzdz'ds \\
&\leq \frac{C_3}{F_T^3}\int_{\mathbb{R}^{2d}} \frac{|\widehat{\varphi}(z)|}{|z|^\alpha |z'|^{2\alpha}} \int_0^T (1-e^{-s|z'|^\alpha})^2 dsdzdz' \\
&\leq \frac{C_3 T}{F_T^3}\int_{\mathbb{R}^{2d}} \frac{|\widehat{\varphi}(z)|}{|z|^\alpha |z'|^{2\alpha}} (1-e^{-T|z'|^\alpha})^2 dzdz' \\
&(F_T = T^{(3-d/\alpha)/2}, \text{ let } \quad z' = T^{-1/\alpha} y) \\
&= \frac{C_3}{T^{(3-d/\alpha)/2}} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|}{|z|^\alpha} dz \int_{\mathbb{R}^d} \left(\frac{1-e^{-|y|^\alpha}}{|y|^\alpha} \right)^2 dy \ \to 0 \quad \text{as} \quad T \to \infty
\end{aligned}
$$

(the integral on z is finite since $\alpha < d$ and the integral on y is finite since $d < 2\alpha$).

Similarly, by (3.39),

$$
\begin{aligned}
J_2(T) &\leq \frac{C}{F_T^3}\int_0^T ds \int_0^s du \int_0^u dr \int_0^u dr' \int_0^s du' \int_{\mathbb{R}^d} \mathcal{T}_r\varphi(x)\mathcal{T}_{r'}\varphi(x)\mathcal{T}_{s-u+u'}\varphi(x)dx \\
&= \frac{C_1}{F_T^3}\int_0^T ds \int_0^s du \int_0^u dr \int_0^u dr' \int_0^s du' \\
&\qquad \int_{\mathbb{R}^{2d}} \widehat{\varphi}(z)e^{-r|z|^\alpha}\widehat{\varphi}(z')e^{-(s-u+u')|z'|^\alpha}\overline{\widehat{\varphi}(z+z')}e^{-r'|z+z'|^\alpha}dzdz'
\end{aligned}
$$

(estimating similarly as for $J_1(T)$)

$$
\begin{aligned}
&\leq C_2 \frac{T}{F_T^3}\int_{\mathbb{R}^{2d}} \frac{1}{|z|^\alpha |z'|^{2\alpha}|z+z'|^\alpha}(1-e^{-T|z'|^\alpha})^2 dzdz' \\
&(F_T = T^{(3-d/\alpha)/2}, \quad \text{let} \quad z = T^{-1/\alpha}y, z' = T^{-1/\alpha}y') \\
&= \frac{C_2}{T^{(d/\alpha - 1)/2}} I,
\end{aligned}
$$

where

$$
I = \int_{\mathbb{R}^{2d}} \frac{1}{|y|^\alpha |y+y'|^\alpha} \left(\frac{1-e^{-|y'|^\alpha}}{|y'|^\alpha} \right)^2 dydy'.
$$

Since $d > \alpha$, to show that $J_2(T) \to 0$ as $T \to \infty$ it suffices to prove that $I < \infty$.

Let

$$
f_1(y) = \frac{1}{|y|^\alpha}\mathbb{1}_{\{|y|\leq 1\}}, \ f_2(y) = \frac{1}{|y|^\alpha}\mathbb{1}_{\{|y|>1\}}, \ g(y') = \left(\frac{1-e^{-|y'|^\alpha}}{|y'|^\alpha} \right)^2.
$$

Then $f_1 \in L^1(\mathbb{R}^d)$ (since $d > \alpha$), $f_2 \in L^2(\mathbb{R}^d)$ (since $d < 2\alpha$) and $g \in L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$ (since $d < 2\alpha$). Hence $f_1 * f_1 \in L^1(\mathbb{R}^d), f_2 * f_2 \in L^\infty(\mathbb{R}^d), f_1 * f_2 \in L^2(\mathbb{R}^d)$. Therefore

$$
\begin{aligned}
I &= \int_{\substack{|y|\leq 1 \\ |y+y'|\leq 1}} + \int_{\substack{|y|>1 \\ |y+y'|>1}} + \int_{\substack{|y|\leq 1 \\ |y+y'|>1}} + \int_{\substack{|y|>1 \\ |y+y'|\leq 1}} \\
&= \int (f_1 * f_1)g + \int (f_2 * f_2)g + \int (f_1 * f_2)g + \int (f_2 * f_1)g < \infty.
\end{aligned}
$$

Proof of (3.30):

Again we take $\Phi(x,t) = \varphi(x)\psi(t)$. Then, from (3.27) and (3.21),

$$\begin{aligned} I_3(T) &\leq \frac{C}{F_T^2}\int_0^T \int_{\mathbb{R}^d} \varphi(x) \int_0^s \mathcal{T}_u\varphi(x) du dx ds \\ &= \frac{C_1}{T^{3-d/\alpha}} \int_0^T \int_0^s \int_{\mathbb{R}^d} |\widehat{\varphi}(z)|^2 e^{-u|z|^\alpha} dz du ds \\ &\leq \frac{C_1}{T^{3-d/\alpha}} \int_0^T \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^\alpha}{|z|^\alpha} dz ds \\ &\leq \frac{C}{T^{2-d/\alpha}} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^\alpha} dz \to 0 \quad \text{as} \quad T \to \infty, \end{aligned}$$

since $\alpha < d < 2\alpha$.

Theorem 2.2. is proved.

The proof of Theorem 2.1 is analogous to that of Theorem 2.2 but much simpler because the equations for $w(x,r,t)$, etc. are linear in this case. The covariance function (1.1) of fBm appears thanks to the equality

$$\int_0^1 \int_v^1 (u-v)^{-d/\alpha} \chi(u)\chi(v) du dv = C \int_0^1 \int_0^u (u^h + v^h + (u-v)^h)\psi(u)\psi(v) dv du,$$

which is a counterpart of (3.34) (recall that $h = 2 - d/\alpha$ in this case).

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