On weighted approximations in $D[0,1]$ with applications to self-normalized partial sum processes

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ABSTRACT

Let $X, X_1, X_2, ...$ be a sequence of non-degenerate i.i.d. random variables with mean zero. The best possible weighted approximations in $D[0,1]$ for the partial sum processes $\{S_{[nt]}, 0 \le t \le 1\}$, where $S_n = \sum_{j=1}^n X_j$, are investigated under the assumption that X belongs to the domain of attraction of the normal law. The results are used to establish similar results for the sequence of self-normalized partial sum processes $\{S_{[nt]}/V_n, 0 \le t \le 1\}$, where $V_n^2 = \sum_{j=1}^n X_j^2$. L_p approximations of self-normalized partial sum processes are also discussed.

Key Words and Phrases: Weighted approximations, Self-normalized sums, Domain of attraction of the normal law, L_p approximations.

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1 Introduction and main results

Let $X, X_1, X_2, ...$ be a sequence of non-degenerate i.i.d. random variables with $EX = 0$ and let $S_n = \sum_{j=1}^n X_j, n \geq 1$, denote their partial sums. The classical weak invariance principle in probability states that, on an appropriate probability space, as $n \to \infty$,

$$\sup_{0\leq t\leq 1} \Big| \frac{S_{[nt]}}{\sqrt{n}\sigma} - \frac{W(nt)}{\sqrt{n}} \Big| = o_P(1) \quad \text{if and only if} \quad Var(X) = \sigma^2 < \infty, \tag{1}$$

where $\{W(t), 0 \leq t < \infty\}$ is a standard Wiener process. This invariance principle in probability is a stronger version of Donsker's classical Functional Central Limit Theorem. The normalizer $(n\sigma^2)^{-1/2}$ in (1) is that of the classical central limit theorem when $Var(X) < \infty$.

In view of (1), it is natural to seek conditions for having the following weighted approximation:

$$\sup_{0<t\leq 1} \Big| \frac{S_{[nt]}}{\sqrt{n}\sigma} - \frac{W(nt)}{\sqrt{n}} \Big| \Big/ q(t) \quad = \quad o_P(1), \quad n \to \infty, \tag{2}$$

where $q(t)$ is a non-negative function on $(0, 1]$ approaching zero as $t \downarrow 0$. In this direction, using the methods developed by Csörgő, Csörgő, Horváth and Mason [CsCsHM] (1986), and subsequently by Csörgő and Horváth (1986), for weighted empirical and quantile processes, together with the Komlós, Major and Tusnády [KMT] (1976) and Major (1976) strong approximations for partial sum processes, Csörgő and Horváth (1988) concluded that, on an appropriate probability space, as $n \to \infty$,

$$\sup_{1/n\leq t\leq 1} n^{\mu} \Big| \frac{S_{[nt]}}{\sqrt{n}\sigma} - \frac{W(nt)}{\sqrt{n}} \Big| \Big/ t^{1/2-\mu} \quad = \quad O_P(1), \tag{3}$$

for any $0 \leq \mu \leq 1/2 - 1/r$, if we assume $E|X|^r < \infty$ for some $r > 2$.

Let Q be the class of positive functions $q(t)$ on $(0, 1]$, i.e., $\inf_{\delta\leq t\leq 1} q(t) > 0$ for $0 < \delta < 1$, which are nondecreasing near zero, and let

$$I(q, c) = \int_{0+}^{1} t^{-1} \exp\Big(-cq^2(t)/t\Big)\, dt, \quad 0 < c < \infty.$$

By virtue of the result (3), we have the following fact: if $E|X|^r < \infty$ for any $r > 2$ and $q \in Q$ such that $I(q, c) < \infty$ for any $c > 0$, then the claim (2) holds true. We mention that $q \in Q$ such that $I(q, c) < \infty$ for any $c > 0$ is the the optimal class of weight functions as in CsCsHM (1986) to make (2) true. For more details along these lines, we refer to Szyszkowicz

(1991, 1992, 1996, 1997) where, by a different method, the result (2) is also established for the optimal class of weight functions assuming only the existence of two moments.

In this paper, we establish a result which is similar to (2) only under the assumption that X belongs to the domain of attraction of the normal law, written $X \in$ DAN throughout. In fact, it is well-known that $X \in$ DAN with $EX = 0$ if and only if there exists a sequence of constants $d_n \uparrow \infty$ such that, as $n \to \infty$, $S_n/d_n \to_{\mathcal{D}} N(0,1)$. It is natural to ask whether a similar version of the weighted approximation (2) could also hold true. Throughout the paper, we make use of the notation $l(x) = EX^2 I_{(|X| \le x)}$, $b = \inf\{x \ge 1 : l(x) > 0\}$,

$$\eta_j = \inf\Big\{s : s \ge b+1, \frac{l(s)}{s^2} \le \frac{1}{j}\Big\}, \quad j = 1, 2, \ldots .$$

and $b_n^2 = n\, l(\eta_n)$. We have the following theorem which provides an affirmative answer to this paramount question (cf. Corollary 1).

Theorem 1 *Let $q \in Q$ and assume that $X \in DAN$ and $EX = 0$. Then, on an appropriate probability space for $X, X_1, X_2, \cdots$, a standard Wiener process $\{W(t), 0 \le t < \infty\}$ can be constructed in such a way that the following statements hold true.*

(a) *If $I(q,c) < \infty$ for any $c > 0$, then*

$$\sup_{1/n \le t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \quad = \quad o_P(1). \tag{4}$$

(b) *If $I(q,c) < \infty$ for some $c > 0$, then*

$$\sup_{1/n \le t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \quad = \quad O_P(1). \tag{5}$$

(c) *If $I(q,c) < \infty$ for some $c > 0$, then there exists a sequence of constants $\tau_n \to 0$ such that*

$$\sup_{\tau_n \le t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \quad = \quad o_P(1). \tag{6}$$

Remark 1. If $EX^2 < \infty$, then τ_n in (6) can be changed to $1/n$. In this case, it is readily seen that $n\sigma^2/b_n^2 \to 1$, as $n \to \infty$. Hence, via Theorem 1.1 of Szyszkowicz (1997), a standard Wiener process $\{W(t), 0 \le t < \infty\}$ can be constructed such that

$$\begin{aligned}
&\sup_{1/n \le t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \\
&\quad \le \frac{\sqrt{n}\sigma}{b_n} \sup_{1/n \le t \le 1} \Big| S_{[nt]}/\sqrt{n}\sigma - W(nt)/\sqrt{n} \Big| \Big/ q(t) \\
&\qquad + \Big| \frac{\sqrt{n}\sigma}{b_n} - 1 \Big| \sup_{0 < t \le 1} \Big| n^{-1/2} W(nt) \Big| \Big/ q(t) \\
&\quad = o_P(1),
\end{aligned}$$

where we make use of $\sup_{0<t\le 1} \Big| n^{-1/2} W(nt) \Big| \Big/ q(t) = O_P(1)$, which follows from Lemma 3 below via having $\Big\{ n^{-1/2} W(nt) \Big/ q(t), 0 < t \le 1 \Big\} \stackrel{\mathcal{D}}{=} \Big\{ W(t) \Big/ q(t), 0 < t \le 1 \Big\}$ for each $n \ge 1$. However, it is impossible to replace τ_n in (6) by $1/n$ without further restrictions on X and/or $q(t) \in Q$. In Section 2, an example will be given [cf. Proof of (7)] to show that (6) holds in terms of a standard Wiener process $\{W(t), 0 \le t < \infty\}$, but

$$\sup_{1/n \le t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \ne o_P(1), \tag{7}$$

with $q^2(t) = t \log\log(t^{-1})$, where $\log x = \log(\max\{e, x\})$ here, as well as throughout.

The following corollaries are consequences of Theorem 1, which are also of independent interest.

Corollary 1 *Let $q \in Q$. As $n \to \infty$, the following statements are equivalent:*

(a) *$X \in DAN$ and $EX = 0$;*

(b) *$S_{[nt]}/b_n \Rightarrow_D W(t)$ on $(D[0,1], \| \ /q\|)$ if and only if $I(q,c) < \infty$ for any $c > 0$, where $\{W(t),\ 0 \le t \le 1\}$ is a standard Wiener process and $\| \ /q\|$ is the weighted sup-norm metric in $D[0,1]$ defined by*

$$||(x-y)/q|| = \sup_{0 \le t \le 1} |(x(t) - y(t))/q(t)| \tag{8}$$

whenever this is well defined, i.e., when $\limsup_{t \downarrow 0} |(x(t) - y(t))/q(t)|$ is finite;

(c) *On the probability space of* Theorem 1 *for $X, X_1, X_2, \ldots,$ a standard Wiener process $\{W(t),\ 0 \le t < \infty\}$ can be constructed in such a way that, as $n \to \infty$,*

$$\sup_{0 < t \le 1} |S_{[nt]}/b_n - W(nt)/n^{1/2}|/q(t) = o_P(1) \tag{9}$$

if and only if $I(q,c) < \infty$ for any $c > 0$.

While Corollary 1 is a direct consequence of Theorem 1, we now state convergence in distribution results for sup-functionals of weighted normalized partial sums for the optimal class of weight functions $q \in Q$ satisfying $\limsup_{t\downarrow 0} |W(t)|/q(t) < \infty$ *a.s.* (see Lemma 3). Consequently, the results that follow are not implied by Corollary 1 above, and they cannot be obtained via classical methods of weak convergence either, for tightness in our weighted sup-norm is not guaranteed by Lemma 3.

Corollary 2 *Let $\{W(t), 0 \le t < \infty\}$ be a standard Wiener process, $X \in DAN$ and $EX = 0$.*

(a) *If $q \in Q$, then there exists a sequence of positive constants $\tau_n \to 0$ such that, as $n \to \infty$,*

$$b_n^{-1} \sup_{\tau_n \le t \le 1} |S_{[nt]}|/q(t) \ \to_D \ \sup_{0<t\le 1} |W(t)|/q(t) \tag{10}$$

if and only if $I(q,c) < \infty$ for some $c > 0$.

(b) *If $q \in Q$ and $q(t)$ is nondecreasing on $(0,1]$, then as $n \to \infty$,*

$$b_n^{-1} \sup_{0<t\le 1} |S_{[nt]}|/q(t) \ \to_D \ \sup_{0<t\le 1} |W(t)|/q(t) \tag{11}$$

if and only if $I(q,c) < \infty$ for some $c > 0$. Consequently, as $n \to \infty$, we have

$$b_n^{-1} \sup_{0<t\le 1} |S_{[nt]}|/(t \log\log(t^{-1}))^{1/2} \ \to_D \ \sup_{0<t\le 1} |W(t)|/(t \log\log(t^{-1}))^{1/2}.$$

Remark 2. Corollary 2 rhymes with Corollary 1.1 of Szyszkowicz (1997), where X is assumed to have two moments. It remains an open problem whether $\sup_{\tau_n \le t \le 1}$ in (10) can be improved to $\sup_{0<t\le 1}$ without further restrictions on X and/or $q(t) \in Q$.

Remark 3. The main results of Corollary 1 were announced without proofs in Csörgő, Szyszkowicz and Wang [CsSzW] (2004), where we reviewed weighted approximations and strong limit theorems for self-normalized partial sum processes. It is interesting and also of interest to note that the class of the weight functions in Corollary 2 is bigger than that in Corollary 1. Such a phenomenon was first noticed and proved for weighted empirical and quantile processes by CsCsHM (1986) and then by Csörgő and Horváth (1988) for partial sums on assuming $E|X|^v < \infty$ for some $v > 2$. For more details along these lines, we refer to Szyszkowicz (1991, 1996, 1997), and to Csörgő, Norvaiša and Szyszkowicz (1999).

We next consider applications of Theorem 1 to the so-called self-normalized partial sum processes defined by $\{S_{[nt]}/V_n, 0 \le t < \infty\}$, where $V_n^2 = \sum_{j=1}^n X_j^2$. Note that

$$V_n^2/b_n^2 \quad \to_P \quad 1 \tag{12}$$

if $X \in$ DAN (see the result (18) in CsSzW (2003), for example). This, together with Theorem 1, yields the following Theorem 2. We mention that Theorem 2 and its corollaries were announced without proofs in CsSzW (2004). We restate them here for convenient reference and further use in the sequel.

Theorem 2 *Assume that $X \in DAN$ and $EX = 0$. Then, on the probability space of* Theorem 1 *for $X, X_1, X_2, \cdots$, the there constructed standard Wiener process $\{W(t), 0 \le t < \infty\}$ is such that that the following statements hold true.*

(a) *Let $q \in Q$. Then, as $n \to \infty$,*

$$\sup_{1/n \le t \le 1} \Big| S_{[nt]}/V_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) = \begin{cases} o_P(1) & \text{if } I(q,c) < \infty \text{ for any } c > 0, \\ O_P(1) & \text{if } I(q,c) < \infty \text{ for some } c > 0. \end{cases} \tag{13}$$

(b) *Let $q \in Q$ and $I(q,c) < \infty$ for some $c > 0$. Then there exists a sequence of constants $\tau_n \to 0$ such that as $n \to \infty$,*

$$\sup_{\tau_n \le t \le 1} \Big| S_{[nt]}/V_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) = o_P(1). \tag{14}$$

Corollary 3 *Let $q \in Q$. As $n \to \infty$, the following statements are equivalent:*

(a) *$X \in DAN$ and $EX = 0$;*

(b) *$S_{[nt]}/V_n \Rightarrow_D W(t)$ on $(D[0,1], \| \ /q\|)$ if and only if $I(q,c) < \infty$ for any $c > 0$, where $\| \ /q\|$ is defined as in* Corollary 1 *and $\{W(t),\ 0 \le t \le 1\}$ is a standard Wiener process;*

(c) *On the probability space of* Theorem 1 *for $X, X_1, X_2, \ldots$, a standard Wiener process $\{W(t),\ 0 \le t < \infty\}$ can be constructed in such a way that, as $n \to \infty$,*

$$\sup_{0 < t \le 1} |S_{[nt]}/V_n - W(nt)/n^{1/2}|/q(t) = o_P(1) \tag{15}$$

if and only if $I(q,c) < \infty$ for any $c > 0$.

Corollary 4 *The conclusions of* Corollary 2 *continue to hold true if we replace the normalizing constants b_n by V_n.*

Remark 4. Corollary 3 extends Theorem 1 of CsSzW (2003) where we established an variance principle in probability version of Donsker's theorem for self-normalized partial sum processes. Corollaries 3 and 4 provide basic tools for investigating the limit behavior of statistics that arise in studying the problem of a change in the mean in the domain of attraction of the normal law. For details we refer to Section 5 in CsSzW (2004).

For the sake of Studentized versions of Theorem 2, Corollaries 3 and 4, consider now the sequence $T_{n,t}(.)$ of Student processes in $t \in [0,1]$ on $D[0,1]$, defined as

$$\begin{aligned}\{T_{n,t}(X), 0 \leq t \leq 1\} &:= \Big\{\frac{(1/\sqrt{n})\sum_{i=1}^{[nt]} X_i}{\sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}},\ 0 \leq t \leq 1\Big\} \\ &= \Big\{\frac{\sum_{i=1}^{[nt]} X_i/V_n}{\sqrt{[n-(S_n/V_n)^2]/(n-1)}},\ 0 \leq t \leq 1\Big\}.\end{aligned}$$

Clearly, $T_{n,1}(X)$ is the familiar form of the classical Student ratio. When $X =_D N(\mu, \sigma^2)$, then $T_{n,1}(X-\mu)$ is his famous t-random variable with $n-1$ degrees of freedom [cf. "Student" (1908)]. Clearly, if $T_{n,1}(X)$ or S_n/V_n has an asymptotic distribution, then so does the other, and it is well known that they coincide (cf., e.g., Proposition 1 in Griffin (2002)). Hence, usually, and without loss of generality, the limiting distribution of S_n/V_n is studied in lieu of that of $T_{n,1}(X)$. In fact via Theorem 2 and Corollaries 3 and 4 we follow a functional version of this route. Backtracking a bit, Logan, Mallows, Rice and Shepp (1973) conjectured that "S_n/V_n is asymptotically normal if (and perhaps only if) X is in the domain of attraction of the normal law". It is the "only if" part that has remained open until 1997 for the general case of not necessarily symmetric random variables, when Gine, Götze and Mason (1997) proved the following result.

Theorem A. *The following two statements are equivalent:*

(i) *X is in the domain of attraction of the normal law;*

(ii) *There exists a finite constant μ such that, as $n \to \infty$, $T_{n,1}(X-\mu) \to_D N(0,1)$.*

Moreover, if either (i) *or* (ii) *holds, then $\mu = EX$.*

Furthermore, Chistyakov and Götze (2004) confirmed a second conjecture of Logan et al. (1973) that the Student t-statistic has a non-trivial limiting distribution if and only if X is in the domain of attraction of a stable law with some exponent $\alpha \in (0, 2]$. Our next corollary constitutes a Studentized version of the self-normalized results of Theorem 2, and Corollaries 3 and 4. Thus it amounts to various optimal functional extensions of Theorem A in $(D[0,1], \|/q\|)$.

Corollary 5 *The results* (a), (b) *of* Theorem 2 *continue to hold true when* $EX = \mu$*, with* $T_{n,t}(X - \mu)$ *replacing* $S_{[nt]}/V_n$ *in each of the two statements. Mutatis mutandis, the same is true as well in case of* Corollaries 3 *and* 4.

This paper is organized as follows. In Section 2 we provide proofs of the main results. The proof of Proposition 1, which is used in the proofs of main results, will be given in Section 3. Finally, in Section 4 we will discuss weighted approximations of self-normalized partial sum processes under the optimal class of weight functions in L_p. Throughout the paper, $A, A_1, A_2, ...$ denote constants which may be different at each appearance.

In concluding this section, we mention some of the previous results on weighted approximation. The study of weighted approximations originates from the earlier works of Anderson and Darling (1952), Rényi (1953), Chibisov (1964), Pyke and Shorack (1968) and O'Reilly (1974), who investigated the asymptotic behaviour of weighted empirical and quantile processes. The works of these authors were later extented by CsCsHM (1986), Szyszkowicz (1991, 1992, 1996, 1997) and Csörgő, Norvaiša and Szyszkowicz [CsNSz] (1999). In CsCsHM (1986), the authors established approximations of empirical and quantile processes by sequences of Brownian bridges in weghted supremum metrics for the optimal class of weight functions. Szyszkowicz (1996, 1997) derived similarly optimal weighted approximations of standardized partial sum processes in $D[0, \infty)$ under finite second moment conditions (cf. also Szyszkowicz (1991) concerning related results in $D[0, 1]$). The paper CsNSz (1999) extends the results of Szyszkowicz (1991, 1992, 1996, 1997) to arbitrary positive weight functions q on $(0, \infty]$ for which $\lim_{t\downarrow 0, t\uparrow 0} |W(t)|/q(t) = 0\, a.s.$, or $\limsup_{t\downarrow 0, t\uparrow 0} |W(t)|/q(t) < \infty\, a.s.$ For further results on weighted approximations for empirical, quantile and standardized partial sum processes with applications to change-point analysis, we refer to two books of Csörgő and Horváth (1993, 1997) for details.

2 Proofs of the main results

We first list several lemmas of independent interest that will be used in the proofs of the main results of Section 1. The first result is due to Sakhanenko (1980, 1984, 1985).

Lemma 1 *Let $X_1, X_2, \ldots$ be independent random variables with $EX_j = 0$ and $\sigma_j^2 = EX_j^2 < \infty$ for each $j \geq 1$. Then we can redefine $\{X_j, j \geq 1\}$ on a richer probability space together with a sequence of independent $N(0,1)$ random variables, Y_j, $j \geq 1$, such that for every $p > 2$ and $x > 0$,*

$$P\Big\{ \max_{i \leq n} \Big| \sum_{j=1}^{i} X_j - \sum_{j=1}^{i} \sigma_j Y_j \Big| \geq x \Big\} \leq (Ap)^p x^{-p} \sum_{j=1}^{n} E|X_j|^p,$$

where A is an absolute positive constant.

The next two lemmas are due to CsCsHM (1986) [cf. Lemma A.5.1 and Theorem A.5.1 respectively in Csörgő and Horváth (1997)]. Proofs of Lemmas 2 and 3 can also be found in Section 4.1 of Csörgő and Horváth (1993). For related further results along these lines we refer to Csörgő, Shao and Szyszkowicz (1991) and, for some historical references, to Section 2 of CsNSz (1999) (see the statements (2.5), (2.9) and their discussion in there).

Lemma 2 *Let $q(t) \in Q$. If $I(q,c) < \infty$ for some $c > 0$, then $\lim_{t \downarrow 0} t^{1/2}/q(t) = 0$.*

Lemma 3 *Let $\{W(t), 0 \leq t < \infty\}$ be a standard Wiener process and $q(t) \in Q$. Then,*

(a) *$I(q,c) < \infty$ for any $c > 0$ if and only if $\limsup_{t \downarrow 0} |W(t)|/q(t) = 0$, a.s.*

(b) *$I(q,c) < \infty$ for some $c > 0$ if and only if $\limsup_{t \downarrow 0} |W(t)|/q(t) < \infty$, a.s.*

We are now ready to prove the main results of Section 1.

Proof of Theorem 1. In addition to the notation in Section 1, write

$$X_j^* = X_j I_{(|X_j| \leq \eta_j)} \qquad \text{and} \qquad S_n^* = \sum_{j=1}^{n} X_j^*.$$

By Lemma 1, we can redefine $\{X_j, j \geq 1\}$ on a richer probability space together with a sequence of independent $N(0,1)$ random variables, Y_j, $j \geq 1$, such that for any $x > 0$ and any constant sequence $c_j, j \geq 1$,

$$P\Big\{ \max_{i \leq n} \Big| \sum_{j=1}^{i} c_j \left(X_j^* - EX_j^* \right) - \sum_{j=1}^{i} c_j \sigma_j^* Y_j \Big| \geq x \Big\} \ \leq \ Ax^{-3} \sum_{j=1}^{n} |c_j|^3 E|X|^3 I_{(|X| \leq \eta_j)}, \tag{16}$$

where $\sigma_j^{*2} = Var(X_j^*)$. Let $\{W(t), 0 \le t < \infty\}$ be a standard Wiener process such that

$$W(n) = \sum_{j=1}^{n} Y_j, \qquad n = 1, 2, 3, ...$$

The results (4)-(6) will be shown to hold true for the this way constructed Wiener process, and then Theorem 1 follows accordingly. To prove (4)-(6), we need the following proposition.

Proposition 1 *We have, as $n \to \infty$,*

$$\frac{1}{n}\sum_{j=1}^{n}\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1\Big)^2 \;\to\; 0, \tag{17}$$

$$b_n^{-1}\sup_{0\le t\le 1}\Big|S_{[nt]} - \sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big| \;=\; o_P(1), \tag{18}$$

$$b_n^{-1}\sup_{0< t\le 1}\Big|S_{[nt]} - \sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big|\Big/t^{1/2} \;=\; O_P(1). \tag{19}$$

The proof of Proposition 1 will be given in Section 4. By virtue of Proposition 1, we have that if $I(q, c) < \infty$ for some $c > 0$, then

$$I_n \;:=\; b_n^{-1}\sup_{1/n\le t\le 1}\Big|S_{[nt]} - \sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big|\Big/q(t) \;=\; o_P(1). \tag{20}$$

Indeed it follows from (18) and (19) that, for any $0 < \delta < 1$,

$$\begin{aligned} I_n \;\le\; & \sup_{1/n\le t\le\delta} t^{1/2}/q(t)\; b_n^{-1}\sup_{0<t\le 1}\Big|S_{[nt]} - \sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big|\Big/t^{1/2} \\ & + \sup_{\delta<t\le 1} q^{-1}(t)\; b_n^{-1}\sup_{\delta<t\le 1}\Big|S_{[nt]} - \sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big| \\ =\; & O_P(1)\sup_{1/n\le t\le\delta} t^{1/2}/q(t) + o_P(1)\sup_{\delta<t\le 1} q^{-1}(t). \end{aligned}$$

The claim (20) now follows from Lemma 2 and the fact that $\inf_{\delta\le t\le 1} q(t) > 0$ for any $\delta > 0$.

We now proceed to prove (4)-(6). Consider (4) and (5) first. We have

$$\sup_{1/n\le t\le 1}\Big|S_{[nt]}/b_n - W(nt)/\sqrt{n}\Big|\Big/q(t) \;\le\; I_n + I_1(n) + I_2(n), \tag{21}$$

where $I_1(n) = n^{-1/2}\sup_{1/n\le t\le 1}\Big|W([nt]) - W(nt)\Big|\Big/q(t)$ and

$$I_2(n) = \sup_{1/n\le t\le 1}\Big|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j - n^{-1/2}W([nt])\Big|\Big/q(t).$$

Note that $\left\{\frac{W(nt)-W([nt])}{\sqrt{nt}}, 0<t\le 1\right\} \stackrel{\mathcal{D}}{=} \left\{W(\frac{nt-[nt]}{nt}), 0<t\le 1\right\}$ and $\frac{|nt-[nt]|}{nt} \le 1$ for $1/n \le t \le 1$. Similarly to the proof of (20), it follows that

$$\begin{aligned} I_1(n) &\le \sup_{1/n\le t\le\delta} t^{1/2}/q(t) \sup_{1/n\le t\le\delta} \Big|W([nt])-W(nt)\Big|\Big/\sqrt{nt} \\ &\qquad + n^{-1/2} \sup_{\delta\le t\le 1} q^{-1}(t) \sup_{\delta\le t\le 1}\Big|W([nt])-W(nt)\Big| \\ &= o_P(1), \end{aligned} \tag{22}$$

whenever $I(q,c)<\infty$ for some $c>0$. As to $I_2(n)$, we have

$$I_2(n) \le \sup_{1/n\le t\le\delta}\Big|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j - n^{-1/2}W([nt])\Big|\Big/q(t) + II(n)\sup_{\delta\le t\le 1} q^{-1}(t), \tag{23}$$

for any $\delta\in(0,1)$, where $II(n) = \max_{1\le k\le n}\Big|b_n^{-1}\sum_{j=1}^k \sigma_j^* Y_j - n^{-1/2}\sum_{j=1}^k Y_j\Big|$. Furthermore, $II(n)=o_P(1)$, since it follows from (17) that, as $n\to\infty$,

$$E[II(n)]^2 \le \frac{1}{n} E\max_{1\le k\le n}\Big|\sum_{j=1}^k\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)Y_j\Big|^2 \le \frac{1}{n}\sum_{j=1}^n\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)^2 \to 0.$$

In view of (20)-(23), the results (4) and (5) will follow if we prove

$$\begin{aligned} I_2^{(1)}(n,\delta) &:= \sup_{1/n\le t\le\delta}\Big|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j - n^{-1/2}W([nt])\Big|\Big/q(t) \\ &= \begin{cases} o_P(1) & \text{if } I(q,c)<\infty \text{ for any } c>0, \\ O_P(1) & \text{if } I(q,c)<\infty \text{ for some } c>0, \end{cases} \end{aligned} \tag{24}$$

by letting $n\to\infty$ first and then $\delta\downarrow 0$. In order to prove (24), let $\delta>0$ be small enough, so that $q(t)$ is already nondecreasing on $(0,\delta)$ and let n be such that $1/n<\delta$. Since $\sigma_j^* \le \left(EX^2 I_{(|X|\le\eta_j)}\right)^{1/2} \le l^{1/2}(\eta_n), 1\le j\le n$, we have $\frac{1}{n}\sum_{j=1}^{[nt]}\left(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\right)^2 \le t$, for $t\in[0,1]$. On the other hand, for each $n\ge 1$,

$$\left\{n^{-1/2}\sum_{j=1}^{[nt]}\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)Y_j, 0<t\le 1\right\} \stackrel{\mathcal{D}}{=} \left\{W\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)^2\Big), 0<t\le 1\right\}.$$

Now it is readily seen that

$$\begin{aligned} I_2^{(1)}(n,\delta) &\stackrel{\mathcal{D}}{=} \sup_{1/n\le t\le\delta}\Big|W\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)^2\Big)\Big|\Big/q(t) \\ &\le \sup_{1/n\le t\le\delta}\Big|W\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)^2\Big)\Big|\Big/q\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)}-1\Big)^2\Big) \\ &\le \sup_{0<t\le\delta}\Big|W(t)|/q(t). \end{aligned} \tag{25}$$

This, together with Lemma 3, implies (24). The proofs of (4) and (5) is now complete.

We next prove (6). Similarly to the proofs of (4) and (5), it suffices to show that there exists a sequence of positive constants $\tau_n \to 0$ such that if $I(q,c) < \infty$ for some $c > 0$, then

$$\sup_{\tau_n \le t \le \delta} \Big| b_n^{-1} \sum_{j=1}^{[nt]} \sigma_j^* Y_j - n^{-1/2} W([nt]) \Big| \Big/ q(t) = o_P(1), \tag{26}$$

when $n \to \infty$ first and then $\delta \downarrow 0$. In fact, in view of (17), there exists a sequence of positive constants $\kappa_n \to \infty$ so that

$$\frac{\kappa_n}{n} \sum_{j=1}^{n} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \to 0, \quad \text{as } n \to \infty.$$

This implies that for any $\epsilon > 0$,

$$\frac{1}{nt} \sum_{j=1}^{[nt]} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \le \frac{\kappa_n}{n} \sum_{j=1}^{n} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \le \epsilon^2,$$

whenever $1/\kappa_n \le t \le 1$ and n is large enough. Let $\tau_n = 1/\kappa_n$ and δ be small enough so that $q(t)$ is nondecreasing on $(0, \delta)$. Then $\tau_n \to 0$ and, similarly to the proof of (25), we have that if $I(q,c) < \infty$ for some $c > 0$, then

$$\begin{aligned}
& \sup_{\tau_n \le t \le \delta} \Big| n^{-1/2} \sum_{j=1}^{[nt]} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big) Y_j \Big| \Big/ q(t) \\
\stackrel{\mathcal{D}}{=}\; & \sup_{\tau_n \le t \le \delta} \Big| W\Big(\frac{1}{n} \sum_{j=1}^{[nt]} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \Big) \Big| \Big/ q(t) \\
\stackrel{\mathcal{D}}{=}\; & \sup_{\tau_n \le t \le \delta} \frac{\Big| \epsilon\, W\Big(\frac{1}{n\epsilon^2} \sum_{j=1}^{[nt]} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \Big) \Big|}{q\Big(\frac{1}{n\epsilon^2} \sum_{j=1}^{[nt]} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \Big)} \, \frac{q\Big(\frac{1}{n\epsilon^2} \sum_{j=1}^{[nt]} \Big(\frac{\sigma_j^*}{l^{1/2}(\eta_n)} - 1 \Big)^2 \Big)}{q(t)} \\
\le\; & \epsilon \sup_{0 < t \le \delta} \big| W(t) \big| / q(t) = \epsilon\, O_P(1).
\end{aligned} \tag{27}$$

This yields (26) and hence the proof of (6). The proof of Theorem 1 is now complete.

Proof of Corollary 1. We only show part (c) following from part (a). Clearly, (c) implies (b), and that, in turn, (a). In fact, under the conditions that $X \in \mathrm{DAN}$ and $EX = 0$, it follows from Theorem 1 and Lemma 3 that

$$\begin{aligned}
& \sup_{0 < t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \\
& \quad \le \sup_{0 < t < 1/n} \Big| n^{-1/2} W(nt) \Big| \Big/ q(t) + \sup_{1/n \le t \le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) \\
& \quad = o_P(1),
\end{aligned}$$

if $I(q,c)<\infty$ for any $c>0$. This shows sufficiency of part (c). Noting that $\sup_{0<t\le 1}\Big|S_{[nt]}/V_n-W(nt)/\sqrt{n}\Big|\Big/q(t)\ge \sup_{0<t<1/n}\Big|n^{1/2}W(nt)\Big|\Big/q(t)$, the necessity of part (c) follow as in Szyszkowicz (1996) (pages 331-332). Hence we omit these details. The proof of Corollary 1 is now complete.

Proof of Corollary 2. If one of (10) and (11) holds, then

$$\sup_{0<t\le 1}\Big|W(t)/q(t)\Big|<\infty \quad a.s., \tag{28}$$

for any standard Wiener process. Consequently, Lemma 3 implies that $I(q,c)<\infty$ for some $c>0$.

Next we assume $I(q,c)<\infty$ for some $c>0$. By Lemma 3 again, we get (28). Then, clearly as $n\to\infty$, we have that

$$\sup_{\tau_n\le t\le 1}|W(t)|/q(t)\to_D \sup_{0<t\le 1}|W(t)|/q(t),$$

which, together with part (c) of Theorem 1, implies (10).

We next prove (11). Recalling (20), we have

$$b_n^{-1}\sup_{0<t\le 1}\Big|S_{[nt]}\Big|\Big/q(t) \;\le\; \sup_{1/n\le t\le 1}\Big|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big|\Big/q(t)+o_P(1). \tag{29}$$

Since $\sigma_j^{*2}\le EX^2I_{(|X|\le\eta_j)}=l(\eta_j)$, it can be easily seen that $b_n^{-2}\sum_{j=1}^{[nt]}\sigma_j^{*2}\le t$ for $t\in(0,1]$. This, together with the fact that $q(t)$ is nondecreasing on $(0,1]$, yields that

$$\begin{aligned}
\sup_{1/n\le t\le 1}\Big|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j\Big|\Big/q(t) &\overset{\mathcal{D}}{=} \sup_{1/n\le t\le 1}\Big|W\Big(b_n^{-2}\sum_{j=1}^{[nt]}\sigma_j^{*2}\Big)\Big|\Big/q(t)\\
&\le \sup_{1/n\le t\le 1}\Big|W\Big(b_n^{-2}\sum_{j=1}^{[nt]}\sigma_j^{*2}\Big)\Big|\Big/q\Big(b_n^{-2}\sum_{j=1}^{[nt]}\sigma_j^{*2}\Big)\\
&\le \sup_{0<t\le 1}\Big|W(t)|/q(t).
\end{aligned} \tag{30}$$

It follows from (29) and (30) that for any $x\ge 0$

$$\lim_{n\to\infty}P\left(b_n^{-1}\sup_{0<t\le 1}\Big|S_{[nt]}\Big|\Big/q(t)\le x\right)\ge P\left(\sup_{0<t\le 1}\Big|W(t)|/q(t)\le x\right).$$

On the other hand, (10) and the fact that $\tau_n>0$ obviously imply that for any $x\ge 0$

$$\begin{aligned}
\lim_{n\to\infty}P\left(b_n^{-1}\sup_{0<t\le 1}\Big|S_{[nt]}\Big|\Big/q(t)\le x\right) &\le \lim_{n\to\infty}P\left(b_n^{-1}\sup_{\tau_n\le t\le 1}\Big|S_{[nt]}\Big|\Big/q(t)\le x\right)\\
&= P\left(\sup_{0<t\le 1}\Big|W(t)|/q(t)\le x\right).
\end{aligned}$$

Therefore, we get the desired (11). This also complete the proof of Corollary 2.

Proof of (7). Let us consider i.i.d. symmetric random variables $X, X_1, X_2, \ldots$ satisfying $EX^2 I_{(|X|\le x)} = 0$, for $x \le 1$, and

$$l(x) = EX^2 I_{(|X|\le x)} \sim \exp\left((\log x)^\alpha\right), \quad \text{for } x > 1, \tag{31}$$

where $0 < \alpha < 1$. Then $l(x)$ is a slowly varying function at ∞. Hence, $EX = 0$ and X is in the domain of attraction of the normal law. Along the proof and the notations of Theorem 1, a standard Wiener process $\{W(t), 0 \le t < \infty\}$ can be constructed such that (6) holds and

$$\begin{aligned}
&\sup_{1/n\le t\le 1} \left|S_{[nt]}/b_n - W(nt)/\sqrt{n}\right| \Big/ q(t) \\
&\quad\ge \sup_{1/n\le t\le 1} \left|b_n^{-1}\sum_{j=1}^{[nt]} \sigma_j^* Y_j - n^{-1/2}W(nt)\right| \Big/ q(t) - I(n) \\
&\quad\ge \sup_{1/n\le t\le 1/\sqrt{n}} \left|b_n^{-1}\sum_{j=1}^{[nt]} \sigma_j^* Y_j - n^{-1/2}W(nt)\right| \Big/ q(t) - I(n) \\
&\quad\ge \sup_{1/n\le t\le 1/\sqrt{n}} \left|n^{-1/2}W(nt)\right| \Big/ q(t) - \sup_{1/n\le t\le 1/\sqrt{n}} \left|b_n^{-1}\sum_{j=1}^{[nt]} \sigma_j^* Y_j\right| \Big/ q(t) - I(n),
\end{aligned}$$

where $I(n)$ is defined as in (20). Noting that $q^2(t) = t\log\log(t^{-1})$ satisfies $I(q,2) < \infty$, $I(n) = o_P(1)$. So, to show (7), it suffices to show that

$$\sup_{1/n\le t\le 1/\sqrt{n}} \left|n^{-1/2}W(nt)\right| \Big/ q(t) \overset{\mathcal{D}}{=} \sup_{1/n\le t\le 1/\sqrt{n}} \left|W(t)\right| \Big/ q(t) \ne o_P(1) \tag{32}$$

with $q^2(t) = t\log\log(t^{-1})$, and if $I(q,c) < \infty$ for some $c > 0$, then

$$\sup_{1/n\le t\le 1/\sqrt{n}} \left|b_n^{-1}\sum_{j=1}^{[nt]} \sigma_j^* Y_j\right| \Big/ q(t) = o_P(1). \tag{33}$$

We first prove (33). For the $l(x)$ defined in (31), it can be easily shown that, for n large enough, $\eta_n \ge \sqrt{n}$ and $\max_{1\le j\le\sqrt{n}} \eta_j \le \left(\sqrt{n}\right)^{3/5}$. Hence,

$$\max_{1\le j\le\sqrt{n}} l(\eta_j)/l(\eta_n) \le 2\exp\left[\left(0.3^\alpha - 0.5^\alpha\right)(\log n)^\alpha\right] \to 0, \quad \text{as } n\to\infty.$$

This, together with $\sigma_j^{*2} \le l(\eta_j)$, yields that, as $n\to\infty$,

$$\frac{1}{tb_n^2}\sum_{j=1}^{[nt]} \sigma_j^{*2} \le \frac{1}{nt}\sum_{j=1}^{[nt]} \frac{l(\eta_j)}{l(\eta_n)} \to 0,$$

whenever $1/n \le t \le \sqrt{n}$. Now, a method that is similar to (27) shows that

$$\sup_{1/n\le t\le 1/\sqrt{n}} \Big| b_n^{-1} \sum_{j=1}^{[nt]} \sigma_j^* Y_j \Big| \Big/ q(t) \stackrel{\mathcal{D}}{=} \sup_{1/n\le t\le 1/\sqrt{n}} \Big| W\Big(b_n^{-2} \sum_{j=1}^{[nt]} \sigma_j^{*2} \Big) \Big| \Big/ q(t) = o_P(1),$$

which implies (33).

Next we prove (32). In fact, if (32) is not true, i.e., if

$$\sup_{1/n\le t\le 1/\sqrt{n}} \Big| W(t) \Big| \Big/ q(t) = o_P(1),$$

then there exists an $0 < \epsilon \le 1/4$ such that

$$P\left(\sup_{1/n\le t\le 1/\sqrt{n}} \Big| W(t) \Big| \Big/ q(t) \le \epsilon^{1/2}/2 \right) \ge 1/2.$$

Hence, following the proof of (4.1.14) in Csörgő and Horváth (1993, page 185), we have, for $q^2(t) = t \log\log(t^{-1})$, as $n \to \infty$,

$$\begin{aligned} 0 \ \leftarrow \ & P\left(\sup_{1/n\le t\le 1/\sqrt{n}} \Big| W(t) \Big| \Big/ q(t) > \epsilon^{1/2}/2 \right) \\ \ge \ & \frac{1}{4} \int_{1/n}^{1/\sqrt{n}} \frac{1}{t} \exp\Big(-2\epsilon q^2(t)/t \Big)\, dt \\ \ge \ & \frac{1}{4} \int_{1/n}^{1/\sqrt{n}} \frac{1}{t} \exp\Big(-2^{-1} \log\log(t^{-1}) \Big)\, dt \\ \ge \ & \frac{1}{2}\Big(1 - \frac{1}{\sqrt{2}}\Big) (\log n)^{1/2} \to \infty. \end{aligned}$$

This is a contradiction, which implies that (32) holds. The proof of (7) is now complete.

Proof of Theorem 2. It follows from (5) that if $I(q, c) < \infty$ for some $c > 0$, then

$$\begin{aligned} b_n^{-1} \sup_{1/n\le t\le 1} |S_{[nt]}| \Big/ q(t) \ \le \ & \sup_{1/n\le t\le 1} n^{-1/2} |W(nt)| \Big/ q(t) \\ & + \sup_{1/n\le t\le 1} \Big| S_{[nt]}/b_n - W(nt)/\sqrt{n} \Big| \Big/ q(t) = O_P(1). \end{aligned}$$

This, together with (12), yields that

$$\sup_{1/n\le t\le 1} \Big| S_{[nt]}/V_n - S_{[nt]}/b_n \Big| \Big/ q(t) \ \le \ \Big| \frac{b_n}{V_n} - 1 \Big| \, b_n^{-1} \sup_{1/n\le t\le 1} |S_{[nt]}| \Big/ q(t) = o_P(1), \qquad (34)$$

whenever $I(q, c) < \infty$ for some $c > 0$. Theorem 2 now follows immediately from (34) and Theorem 1.

Proofs of Corollaries 3-5. In view of Theorem 2, the proofs of Corollaries 3-5 are the same as in the proofs of Corollaries 1 and 2 and hence the details are omitted.

3 Proofs of Proposition 1

We only prove (19). The result (17) can be found in CsSzW (2003, page 1235 there) and the result (18) follows from similar arguments. Let

$$Z_j = X_j^* - EX_j^* - \sigma_j^* Y_j, \quad j = 1, 2, ...,$$

and write $\eta_0 = 0$. Note that $\eta_j^2 \le (j+1) l(\eta_j)$ and $l(\eta_n) = \sum_{k=1}^n EX^2 I_{(\eta_{k-1} < |X| \le \eta_k)}$. It follows from the Shorack and Smythe inequality (cf. Shorack and Weller, 1986, p. 844), (16) with $c_j = 1/j^{1/2}$ that, for any $C > 0$,

$$\begin{aligned}
& P\left(\sup_{1/n \le t \le 1} \Big|\sum_{j=1}^{[nt]} Z_j\Big| \Big/ t^{1/2} \ge C\, b_n\right) \\
& \quad \le P\left(\max_{1 \le k \le n} \Big|(k/n)^{-1/2} \sum_{j=1}^{k} Z_j\Big| \ge C\, b_n\right) \\
& \quad \le P\left(\max_{1 \le k \le n} \Big|\sum_{j=1}^{k} j^{-1/2} Z_j\Big| \ge 2^{-1} C\, b_n/\sqrt{n}\right) \\
& \quad \le A \left(\frac{2\sqrt{n}}{C\, b_n}\right)^3 \sum_{j=1}^{n} j^{-3/2} E|X|^3 I_{(|X| \le \eta_j)} \\
& \quad \le AC^{-3} l^{-3/2}(\eta_n) \sum_{k=1}^{n} k^{-1/2} E|X|^3 I_{(\eta_{k-1} < |X| \le \eta_k)} \\
& \quad \le AC^{-3} l^{-3/2}(\eta_n) \sum_{k=1}^{n} l^{1/2}(\eta_k) E|X|^2 I_{(\eta_{k-1} < |X| \le \eta_k)} \\
& \quad \le AC^{-3}.
\end{aligned}$$

This yields

$$J_{1n} := b_n^{-1} \sup_{1/n \le t \le 1} \Big|\sum_{j=1}^{[nt]} Z_j\Big| \Big/ t^{1/2} = O_P(1). \tag{35}$$

Similarly, by noting that $l(\eta_n) = \sum_{k=1}^n EX^2 I_{(\eta_{k-1} < |X| \le \eta_k)}$ and $E|X| I_{(|X| > \eta_n)} = o(\eta_n^{-1} l(\eta_n)) = o(b_n/n)$ [see Lemma 1 of CsSzW (2003) for example], it follows from (7) in CsSzW (2003) with $a_j = EX^2 I_{(\eta_{j-1} < |X| \le \eta_j)}$ that

$$P\left(\sup_{1/n \le t \le 1} \Big|\sum_{j=1}^{[nt]} (X_j - X_j^* + EX_j^*)\Big| \Big/ t^{1/2} \ge C\, b_n\right)$$

$$
\begin{aligned}
&\le P\left(\max_{1\le k\le n}\sum_{j=1}^{k}\frac{1}{j^{1/2}}\Big(|X_j|I_{(|X_j|>\eta_j)}+E|X_j|I_{(|X_j|>\eta_j)}\Big)\ge Cb_n/n^{1/2}\right)\\
&\le \frac{2n^{1/2}}{Cb_n}\sum_{k=1}^{n}\frac{1}{k^{1/2}}E|X|I_{(|X|>\eta_k)}\\
&\le \frac{2n}{Cb_n}E|X|I_{(|X|>\eta_n)}+\frac{2C^{-1}}{l^{1/2}(\eta_n)}\sum_{k=1}^{n}\frac{1}{k^{1/2}}E|X|I_{(\eta_k<|X|\le\eta_n)}\\
&\le AC^{-1}+\frac{2C^{-1}}{l^{1/2}(\eta_n)}\sum_{k=1}^{n}\eta^{-1/2}(\eta_k)\,E|X|^2I_{(\eta_k<|X|\le\eta_{k+1})}\\
&\le AC^{-1}.
\end{aligned}
$$

This yields

$$
J_{2n} \;:=\; b_n^{-1}\sup_{1/n\le t\le 1}\Big|\sum_{j=1}^{[nt]}(X_j-X_j^*+EX_j^*)\Big|\Big/t^{1/2}=O_P(1). \tag{36}
$$

Combining the estimates (35) and (36), we obtain

$$
b_n^{-1}\sup_{0<t\le 1}\Big|S_{[nt]}-\sum_{j=1}^{[nt]}\sigma_j^*Y_j\Big|\Big/t^{1/2} \;\le\; J_{1n}+J_{2n}=O_P(1),
$$

which implies (19). The proof of Proposition 1 is complete.

4 L_p-approximations of weighted self-normalized processes

Noting that $\limsup_{t\downarrow 0}|W(t)|/t^{1/2}=\infty\,a.s.$, it is impossible to extend Theorem 2 and its corollaries to the weight function $q(t)=t^{1/2}$. However, due to the finiteness of the integral $\int_0^1|W(t)|/t^{1/2}dt$, such a weight function is an immediate candidate for weighted L_1-approximation. We still use Q to denote the class of those positive functions on $(0,1]$ for which $q(t)$ is nondecreasing near zero. The main purpose of this section is to establish L_p-approximations of weighted self-normalized partial sum processes, and thus to extend Theorem 1.1 of Szyszkowicz (1993) that is based on assuming two moments when working with standardized partial sums.

Theorem 3 *Let $q\in Q$ and $0<p<\infty$. Let $EX=0$ and X be in the domain of attraction of the normal law. Then the following statements hold true.*

(a) *On an appropriate probability space for the i.i.d. random variables* $X, X_1, X_2, \ldots$, *we can construct a standard Wiener process* $\{W(t), 0 \le t < \infty\}$ *such that, as* $n \to \infty$,

$$\int_0^1 \left| V_n^{-1} S_{[nt]} - n^{-1/2} W(nt) \right|^p \Big/ q(t) dt = o_P(1) \tag{37}$$

if and only if

$$\int_{0+}^1 t^{p/2} / q(t) dt < \infty. \tag{38}$$

(b) *Let* $\{W(t), 0 \le t < \infty\}$ *be a standard Wiener process. Then, as* $n \to \infty$,

$$\int_0^1 \left| V_n^{-1} S_{[nt]} \right|^p \Big/ q(t) dt \to_D \int_0^1 \left| W(t) \right|^p \Big/ q(t) dt \tag{39}$$

if and only if (38) *holds.*

It is of interest to call attention to the fact that (a) and (b) are equivalent under the conditions of Theorem 3. Thus, unlike in the case of Theorem 2 *versus* having also Corollary 4 in addition to Corollary 3, in L_p we have (a) and (b) for the same class of weight functions.

In the proof of Theorem 3 we make use of the following result, which is a consequence of Corollary 2.1 of Csörgő, Horváth and Shao (1993).

Lemma 4 *Let* $\{W(t), 0 \le t < \infty\}$ *be a standard Wiener process,* $0 < p < \infty$, *and assume that* q *is a positive function on* $(0, 1]$. *Then* (38) *holds if and only if*

$$\int_{0+}^1 |W(t)|^p / q(t) dt < \infty \qquad a.s. \tag{40}$$

We now turn to the proof of Theorem 3.

Proof. We first show (a). If (37) is satisfied, then

$$\int_0^{1/n} \left| n^{-1/2} W(nt) \right|^p \Big/ q(t) dt \stackrel{D}{=} \int_0^{1/n} \left| W(t) \right|^p \Big/ q(t) dt = o_P(1).$$

Therefore, following the proof of Theorem 1.1 of Szyszkowicz (1993), we have (40), and hence also (38), via Lemma 4.

We now prove that (38) implies (37). Still using the notations in the proof of Theorem 1, it can be easily seen that for $\delta \in (0, 1)$,

$$
\begin{aligned}
&\int_0^1 \left|V_n^{-1}S_{[nt]} - n^{-1/2}W(nt)\right|^p \Big/ q(t)dt \\
&\quad \le \int_\delta^1 \left|V_n^{-1}S_{[nt]} - n^{-1/2}W(nt)\right|^p \Big/ q(t)dt \\
&\qquad + \left(\frac{b_n}{V_n}\right)^p \int_{1/n}^\delta \left|b_n^{-1}\left(S_{[nt]} - (S^*_{[nt]} - ES^*_{[nt]})\right)\right|^p \Big/ q(t)dt \\
&\qquad + \left(\frac{b_n}{V_n}\right)^p \int_{1/n}^\delta \left|b_n^{-1}\left(S^*_{[nt]} - ES^*_{[nt]}\right) - b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j\right|^p \Big/ q(t)dt \\
&\qquad + \left(\frac{b_n}{V_n}\right)^p \int_{1/n}^\delta \left|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j\right|^p \Big/ q(t)dt \; + \; \int_0^\delta \left|n^{-1/2}W(nt)\right|^p \Big/ q(t)dt \\
&\quad := II_1(n) + II_2(n) + II_3(n) + II_4(n) + II_5(n). \qquad (41)
\end{aligned}
$$

By Theorem 2, we have

$$
\begin{aligned}
II_1(n) &\le \sup_{0\le t\le 1}\left|V_n^{-1}S_{[nt]} - n^{-1/2}W(nt)\right|^p \int_\delta^1 1\Big/q(t)dt \\
&= o_P(1), \quad \text{for any } \delta \in (0,1). \qquad (42)
\end{aligned}
$$

By (12) and (36), it follows that

$$
II_2(n) \le \left(\frac{b_n}{V_n}\right)^p (J_{2n})^p \int_{1/n}^\delta t^{p/2}\Big/q(t)dt = O_P(1)\int_0^\delta t^{p/2}\Big/q(t)dt. \qquad (43)
$$

Similarly, by (12) and (35), we get

$$
II_3(n) \le \left(\frac{b_n}{V_n}\right)^p (J_{1n})^p \int_{1/n}^\delta t^{p/2}\Big/q(t)dt = O_P(1)\int_0^\delta t^{p/2}\Big/q(t)dt. \qquad (44)
$$

Let δ be small enough so that q is already nondecreasing on $(0,\delta)$. Similarly to the proof of (30), we have

$$
\begin{aligned}
\int_{1/n}^\delta \left|b_n^{-1}\sum_{j=1}^{[nt]}\sigma_j^* Y_j\right| \Big/ q(t)dt &\stackrel{\mathcal{D}}{=} \int_{1/n}^\delta \left|W\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\frac{\sigma_j^{*2}}{l(\eta_n)}\Big)\right| \Big/ q(t)dt \\
&\le \int_{1/n}^\delta \left|W\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\frac{\sigma_j^{*2}}{l(\eta_n)}\Big)\right| \Big/ q\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\frac{\sigma_j^{*2}}{l(\eta_n)}\Big)dt.
\end{aligned}
$$

This, together with (12), implies that

$$
II_4(n) = O_P(1)\int_0^\delta \left|W\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\frac{\sigma_j^{*2}}{l(\eta_n)}\Big)\right| \Big/ q\Big(\frac{1}{n}\sum_{j=1}^{[nt]}\frac{\sigma_j^{*2}}{l(\eta_n)}\Big)dt. \qquad (45)
$$

We also have for each n

$$II_5(n) \overset{\mathcal{D}}{=} \int_0^\delta \left|W(t)\right|^p \Big/ q(t)dt. \tag{46}$$

Taking $n \to \infty$ and then δ arbitrarily small, on using (38) via Lemma 4, by (41)-(46) we arrive at (37).

We next show (b). If (39) holds true, then clearly we have (40) and hence also (38) by Lemma 4. Conversely, we now assume (38). Since q is a positive function on $(0, 1]$, it follows from Theorem 2 that, as $n \to \infty$,

$$\int_\delta^1 \left|V_n^{-1} S_{[nt]}\right|^p \Big/ q(t)dt \to_D \int_\delta^1 \left|W(t)\right|^p \Big/ q(t)dt$$

for all $\delta \in (0, 1)$. On using (38), by Lemma 4 we conclude that

$$\lim_{\delta \downarrow 0} \int_0^\delta \left|W(t)\right|^p \Big/ q(t)dt = 0 \qquad a.s.$$

¿From here on, mutatis mutandis, the proof follows similarly to that of proving (iii) of Theorem 1.1 of Csörgő, Horváth and Shao (1993).

This completes the proof of Theorem 3.

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